AD626407

DISTRIBUTION OF THIS
DOCUMENT IS UNLIMITED.

GEOSCIENCES OPERATIONS
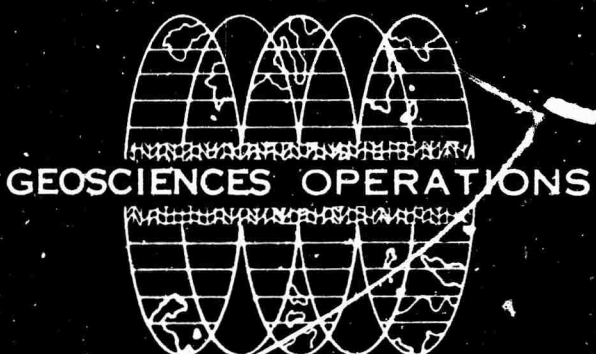
TEXAS INSTRUMENTS
INCORPORATED
SCIENCE SERVICES DIVISION
6000 LEMMON AVENUE
P O BOX 5621 · DALLAS, TEXAS

ARRAY RESEARCH
THEORETICAL CAPABILITY OF SYSTEMS
OF HORIZONTAL SEISMOMETERS FOR PREDICTING
A VERTICAL COMPONENT IN AMBIENT
TRAPPED-MODE NOISE

SPECIAL REPORT NO. 7

By

Thomas F. Potter and Robert B. Roden

Milo M. Backus, Project Scientist

TEXAS INSTRUMENTS INCORPORATED
Science Services Division
P.O. Box 5621
Dallas, Texas 75222

ACKNOWLEDGEMENTS

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

PART I   MULTICOMPONENT SEISMOMETERS

TABLE OF CONTENTS (CONTD)

PART II   MULTICOMPONENT ARRAYS

LIST OF ILLUSTRATIONS

LIST OF TABLES

PART I
MULTICOMPONENT SEISMOMETERS

BLANK PAGE

INTRODUCTION

For enhancing vertically-incident P-waves, the application of multichannel filter techniques to seismometer systems consisting of both horizontal- and vertical-component instruments presents an attractive alternative to the use of systems consisting solely of vertical seismometers. The advantage of using horizontals is that it permits a type of noise-reduction processing which (1) exploits coupling between horizontal and vertical components of ambient seismic noise and (2) preserves the exact form of the P-wave signal.

In this report we shall discuss the theoretical capabilities of systems consisting of a single vertical seismometer v and one or more horizontal seismometers h_1, h_2, . . . h_M. We shall restrict our attention to the following type of processing: we apply to each horizontal seismometer output h_j a linear time-invariant filter g_j, and subtract all of the filter outputs $g_j \circledast h_j$ from v to obtain the output o of our processor. Thus,

$$o = v - g_1 \circledast h_1 - g_2 \circledast h_2 - \cdots g_M \circledast h_M \text{ (Figure 1.1).}$$

Evidently, this processing scheme leads to no degradation of the P-wave signal since the vertically-incident P-wave has no horizontal component. In order to achieve maximum noise suppression, the filters g_j must be designed in such a way that (in the absence of signal)

$$\lim_{T \to \infty} \int_{-T}^{T} \left[o(t) \right]^2 dt$$

is minimized. Such optimal filters are called Wiener prediction filters;
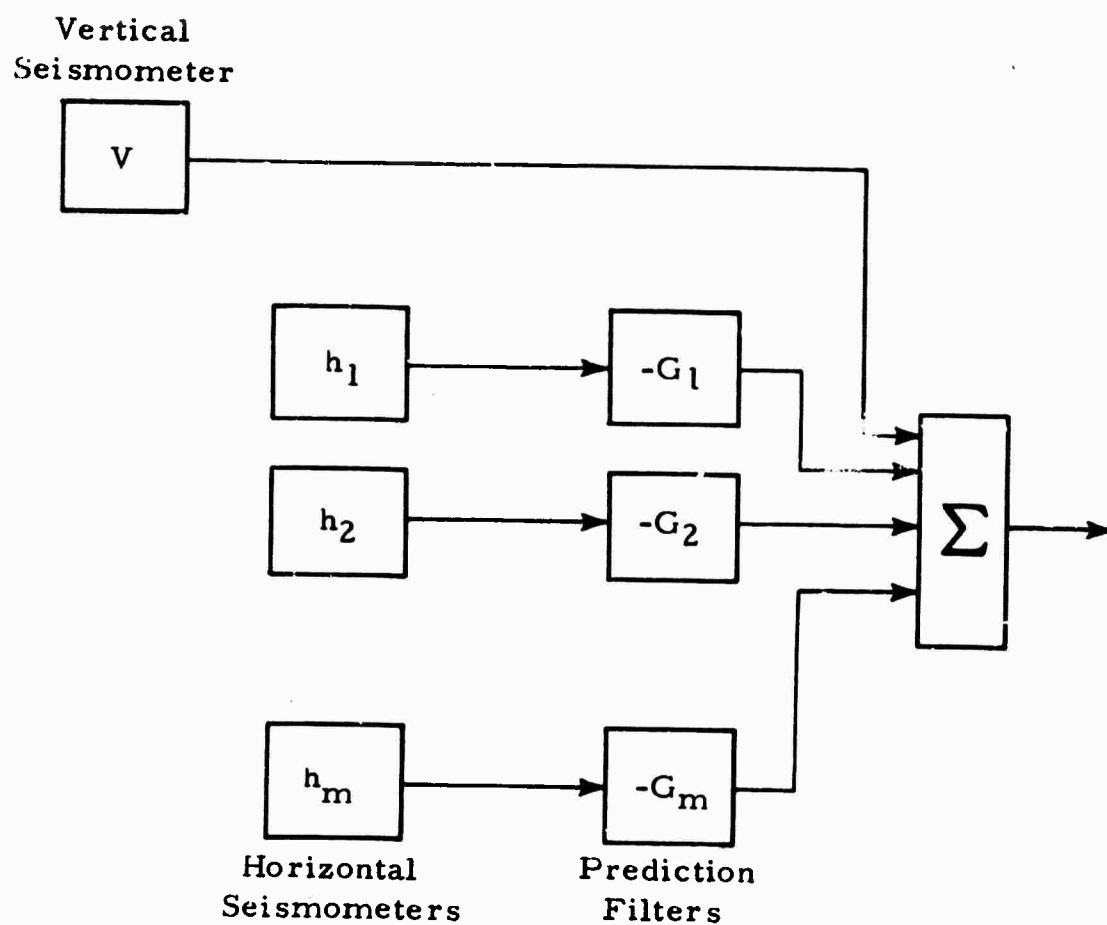
Figure I-1. Block Diagram of Processor for Multicomponent
Seismometer Array

they are determined by the matrix equation (Burg, '964)

$$
\begin{bmatrix}
\phi_{11} & \phi_{21} & \ldots . & \phi_{M1} \\
 & & & \cdot \\
\phi_{12} & \phi_{22} & \ldots & \cdot \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
\cdot & & & \\
\phi_{1M} & . & . & . & . & \phi_{MM}
\end{bmatrix}
\begin{Bmatrix}
G_i \\
G_2 \\
\cdot \\
\cdot \\
\cdot \\
G_M
\end{Bmatrix}
=
\begin{Bmatrix}
\phi_{V1} \\
\phi_{V2} \\
\cdot \\
\cdot \\
\cdot \\
\phi_{VM}
\end{Bmatrix}
$$

where the G_j are the frequency-domain responses of the filters g_j, and the ϕ_{jk}, ϕ_{vk} are the auto- and crosspower spectra of the sei.mometer traces.

   The simplest processor of this type is the 2-component seismometer at point location; i. e., a single vertical seismometer v and a single horizontal seismometer h located at the same point un the earth's surface. Here, the output of the processor is simply

$$o = v - g \circledast h$$

where $g(t)$ is determined by

$$G(f) = \phi_{vh} / \phi_{hh}$$

Part I of this report is devoted to a detailed consideration of the capabilities and limitations of this 2-channel processor in ambient trapped-mode noise. The feasibility of using a 2-component seismometer as a P-wave enhancer is suggested by the observation that complete suppression of noise would be achieved with this system in the ideal situation tha. the noise is simply a unidirectional single-mode Rayleigh wave. In that case, v would be purely a convolution of h; that is, $v = k \circledast h$ for some

I-3

function k(t), and taking $g = k$, we get $o(t) = 0$. One should expect that deviation from this ideal performance would be due principally to the following factors:

(1) If the Rayleigh wave field is not unidirectional, but comes from a distribution of azimuths, there will be less than perfect performance even if all the noise propagates in a single mode. In fact, in isotropic noise, uncorrelated by azimuth, no noise reduction whatsoever can be achieved with the system.

(2) Realistically, v and h must be assumed to contain components of random noise uncorrelated with the Rayleigh waves (e.g., amplifier noise).

(3) The presence on the horizontal of Love (SH) waves, statistically independent from the Rayleigh waves, reduces performance. (One may conceive of Love waves on h as a special case of (2), but it will be seen that it may sometimes be misleading to do so.)

(4) The amplitude gains of seismometers vary slowly with time; this places a limitation on the effectiveness of a system in which the prediction filter, once it has been designed, is left online for an extended period of time.

(5) Several modes of Rayleigh-type noise may be present, each having a different transfer-function relating the vertical and horizontal component.

The quantitative effects of (1) through (5) on the prediction capability of the 2-component seismometer at point location are discussed in Sections III, IV, V, VI, and VII, respectively. Section II is devoted to a summary of communication theory formulae and notation used in the rest of the paper.

I-4

Part II begins in Section IX with a derivation of general formulae for the response, in single-mode noise, of a planar array of seismometers v, h_1, h_2, ..., h_M, where the horizontals h_j may have arbitrary orientations and may be separated from each other or from the vertical v. Certain arrays of this type will be shown to give good performance even in isotropic noise. In Section X, the formulae from Section IX are used to calculate the response in isotropic single-mode noise (with the possible addition of Love waves) of arrays consisting of radially-oriented horizontal seismometers regularly spaced around a ring having the vertical seismometer at its center. It will be shown that such array geometries are optimal in a certain sense.

The sections of this report are, for the most part, independent of each other with the following exceptions: Section II is intended for reference only; Section III should be read first; Section IV should be read before Section V; and Section IX should be read before Section X.

SECTION II

COMMUNICATION THEORY FORMULAS AND NOTATION

Primarily for the purpose of fixing notation, we shall state several formulas to be used in the rest of this paper.

A. CORRELATION FUNCTIONS

Let $x(t)$ be a stationary random process. Then the auto-correlation function $\varphi_{xx}(t)$ of $x(t)$ is defined as

$$\varphi_{xx}(t) = \lim_{T \to \infty} \int_{-T}^{T} x(\tau) x(\tau - t) \, d\tau \tag{2.1}$$

By ergodicity, the autocorrelation function $\varphi_{xx}(t)$ depends only upon the ensemble to which $x(t)$ belongs.

The <u>autopower spectrum</u> $\Phi_{xx}(f)$ of $x(t)$ is the Fourier transform of $\varphi_{xx}(t)$:

$$\Phi_{xx}(f) = \int_{-\infty}^{\infty} \varphi_{xx}(t) e^{-i2\pi ft} \, dt \tag{2.2}$$

Similarly, if $x(t)$ and $y(t)$ are stationary random processes, the <u>crosscorrelation function</u> $\varphi_{xy}(t)$ of x and y is

$$\varphi_{xy}(t) = \lim_{T \to \infty} \int_{-T}^{T} x(\tau) y(\tau - t) \, d\tau \tag{2.3}$$

and the <u>crosspower spectrum</u> $\Phi_{xy}(f)$ of x and y is

$$\Phi_{xy}(f) = \int_{-\infty}^{\infty} \varphi_{xy}(t) e^{-i2\pi ft} \, dt \tag{2.4}$$

The crosspower spectrum is easily seen to satisfy

$$\Phi_{\alpha_1 x_1 + \alpha_2 x_2, y}(f) = \alpha_1 \Phi_{x_1 y}(f) + \alpha_2 \Phi_{x_2 y}$$

and   (2.5)

$$\Phi_{yx}(f) = \Phi_{xy}^{*}(f) \qquad \left[\begin{array}{l} * \text{ denotes complex} \\ \text{conjugation} \end{array} \right]$$

B. LINEAR FILTERS

The output $z(t)$ of the most general linear filter applied to $x(t)$ may be expressed as

$$z(t) = (k \circledast x)(t) = \int_{-\infty}^{\infty} k(\tau) x(t - \tau)\, d\tau \qquad (2.6)$$

where $k(\tau)$ is a function such that

$$\int_{-\infty}^{\infty} |k(\tau)|\, d\tau < \infty$$

The Fourier transform $K(f)$ of $k(\tau)$ is called the transfer function of the filter k.

C. CROSSPOWER SPECTRA OF FILTERED TIME FUNCTIONS

Suppose $x(t)$ and $y(t)$ are stationary random processes, and $g(t)$ and $k(t)$ are linear filters. Let $G(f)$ and $K(f)$ be the transfer functions corresponding to g and k respectively. Then it is easily verified from (2.1)-(2.6) that

$$\Phi_{x, k \circledast y}(f) = K^{*}(f)\, \Phi_{xy}(f)$$

                                                                                          (2.7)

$$\Phi_{g \circledast x, y}(f) = G(f)\, \Phi_{xy}(f)$$

and

$$\Phi_{g \circledast x, k \circledast y}(f) = G(f)\, K^{*}(f)\, \Phi_{xy}(f)$$

In particular,

$$\Phi_{g \circledast x, \, g \circledast x}(f) \;=\; |G(f)|^{2} \, \Phi_{xx}(f) \tag{2.8}$$

D. COHERENCE

Let $x(t)$ and $y(t)$ be stationary random processes. The coherence $C_{xy}(f)$ between x and y is defined as follows:

$$C_{xy}(f) \;=\; \frac{\Phi_{xy}(f)\,\Phi_{yx}(f)}{\Phi_{xx}(f)\,\Phi_{yy}(f)} \;=\; \frac{|\Phi_{xy}(f)|^{2}}{\Phi_{xx}(f)\,\Phi_{yy}(f)} \tag{2.9}$$

It may be shown that $C(f)$ satisfies $0 \le C(f) \le 1$ for all real f.

If g and k are linear filters, it is immediate from (2.7) and (2.9) that

$$C_{g \circledast x, \, k \circledast y}(f) \;=\; C_{xy}(f) \tag{2.10}$$

That is, the coherence between x and y is invariant with respect to transformation of x and y by arbitrary linear filters.

E. PREDICTION

Let $v(t)$ and $h(t)$ be stationary random processes (e.g., vertical and horizontal components of ambient seismic noise at some location). We wish to find the best frequency filter $g(t)$ to apply to $h(t)$ such that the filtered signal $h'(t) = (g \circledast h)(t)$ minimizes the expression

$$\lim_{T \to \infty} \int_{-T}^{T} [v(t) - h'(t)]^{2} \, dt$$

If we let $e(t) = v(t) - h'(t)$, and $G(f) =$ Fourier transform of $g(t)$, our problem is equivalent to finding $G(f)$ such that

$$\int_{-\infty}^{\infty} \Phi_{ee}(f) \, df \qquad (2.11)$$

is minimized.

Using (2.5), we have

$$\Phi_{ee}(f) = \Phi_{v-h', \, v-h'}(f) \qquad (2.12)$$

$$= \Phi_{vv}(f) - \Phi_{vh'}(f) - \Phi_{h'v}(f) + \Phi_{h'h'}(f)$$

$$= \Phi_{vv}(f) - 2\mathrm{Re}\left(\Phi_{h'v}(f)\right) + \Phi_{h'h'}(f)$$

Hence by (2.7),

$$\Phi_{ee}(f) = \Phi_{vv}(f) - 2\mathrm{Re}\left(G(f)\,\Phi_{hv}(f)\right) + |G(f)|^2\, \Phi_{hh}(f) \qquad (2.13)$$

The last expression, the integrand in (2.11), is non-negative for all f. Hence to minimize (2.11), we minimize (2.13) at each f. Letting

$$G(f) = \varphi + i\,\psi$$

in (2.13) and setting

$$\frac{\partial\left(\Phi_{ee}(f)\right)}{\partial \varphi} = \frac{\partial\left(\Phi_{ee}(f)\right)}{\partial \psi} = 0$$

we find

$$\varphi = \frac{\mathrm{Re}\left(\Phi_{hv}(f)\right)}{\Phi_{hh}(f)} \quad , \quad \psi = \frac{-\mathrm{Im}\left(\Phi_{hv}(f)\right)}{\Phi_{hh}(f)}$$

Hence the optimum prediction filter is given by

$$G(f) = \frac{\Phi_{vh}(f)}{\Phi_{hh}(f)} \tag{2.14}$$

Substitution of (2.14) into (2.13) shows that the autopower spectrum of the least-mean-square error is

$$\text{minimal} \quad \Phi_{ee}(f) = \Phi_{vv}(f)\left[1 - \frac{\Phi_{vh}(f)\,\Phi_{hv}(f)}{\Phi_{vv}(f)\,\Phi_{hh}(f)}\right] \tag{2.15}$$

$$= \Phi_{vv}(f)\left[1 - C(f)\right]$$

where $C(f) = $ coherence between h and v.

Because of (2.15), we call $1 - C(f)$ the relative prediction error for the frequency f.

BLANK PAGE

SECTION III
SINGLE MODE RAYLEIGH WAVE FIELD[*]

Let O be the origin of an X-Y coordinate system in the plane of the earth's surface and let Z be a vertical axis through O, positive downward. If Q is any point in the X-Y plane, the azimuth of Q is the angle θ measured counterclockwise from the positive X axis to the line segment OQ.

We suppose P to be the location of a 2-component seismometer consisting of a vertical component oriented downward along Z, and a horizontal component oriented in the positive X-direction.

A. FINITE NUMBER OF DISCRETE UNCORRELATED POINT SOURCES

Suppose we have a finite number N of single-mode Rayleigh wave point sources, located at azimuth θ_1, θ_2, . . ., θ_N. Let $\lambda_n(t)$ be the inline horizontal motion at P due to the nth source; $\lambda_n(t)$ is positive in the direction $-\theta_n$, the direction of propagation. Let $v_n(t)$ be the vertical motion at P due to the nth source.

We make the following basic assumptions:

(1) For each n, $\lambda_n(t)$ is a stationary random process.

(2) The autocorrelation functions $\varphi_{\lambda_n \lambda_n}(t)$ are all the same except for multiplicative constants.

(3) All of the crosscorrelation functions $\varphi_{\lambda_m \lambda_n}$, $m \neq n$, are zero.

(4) There exists a function k(t) with $\int_{-\infty}^{\infty} |k(t)| dt < \infty$ such that $v_n = k \circledast \lambda_n$ for all n, i.e.,

$$v_n(t) = \int_{-\infty}^{\infty} k(s) \, \lambda_n(t-s) \, ds$$

---

[*] In this report, the term "Rayleigh wave" is used to refer to all types of elliptically polarized surface waves and is meant to encompass all surface waves except Love (SH) waves.

Let us define

$$h_n(t) = -\cos\theta_n \lambda_n(t) \tag{3.1}$$

$$h(t) = \sum_{n=1}^{N} h_n(t) \tag{3.2}$$

$$v(t) = \sum_{n=1}^{N} v_n(t)$$

Clearly h_n is the contribution to the output of our horizontal seismometer due to the nth source, and $h(t)$ and $v(t)$ are the outputs of the horizontal and vertical seismometers, respectively.

We proceed to calculate the coherence

$$C = \frac{|\Phi_{vh}|^2}{\Phi_{vv}\Phi_{hh}}$$

between v and h.

Our assumption (3) is equivalent to another assumption:

(3a) $\qquad \Phi_{\lambda_m \lambda_n}(f) = 0, \quad m \neq n$

Our assumption (2) is equivalent to another assumption:

(2a) There exists an autopower spectrum Φ_N, and non-negative constants $p(\theta_n)$, $n = 1, 2, \ldots, N$, with $\displaystyle\sum_{n=1}^{N} p(\theta_n) = 1$, such that

$$\Phi_{\lambda_n \lambda_n}(f) = p(\theta_n)\,\Phi(f), \quad n = 1, 2, \ldots, N \tag{3.3}$$

At this point, because of a limiting process to be carried out later, it is desirable to introduce one additional assumption:

$$\int_{-\infty}^{\infty} \Phi(f)\,df = 1 \tag{3.4}$$

III-2

We have

$$(3.5)$$

$$\Phi_{h_n h_n} = \Phi_{\cos \theta_n \lambda_n, \cos \theta_n \lambda_n} = \cos^2 \theta_n \Phi_{\lambda_n \lambda_n} = \cos^2 \theta_n p(\theta_n) \Phi$$

Also, by assumption (4) and formula (3.1) we get

$$- \cos \theta_n v_n = k \circledast h_n \tag{3.6}$$

Hence by (2.7),

$$- \cos \theta_n \Phi_{v_n h_n} = K \Phi_{h_n h_n} = K \cos^2 \theta_n p(\theta_n) \Phi \tag{3.7}$$

where

$$K(f) = \int_{-\infty}^{\infty} k(t) e^{-i2\pi ft} dt$$

is the Fourier transform of k(t). That is,

$$\Phi_{v_n h_n} = - K \cos \theta_n p(\theta_n) \Phi \tag{3.8}$$

From assumption (4) and (2.7) we get

$$\Phi_{v_n v_n} = \Phi_{k \circledast \lambda_n, \, k \circledast \lambda_n} = |K|^2 \Phi_{\lambda_n \lambda_n} = |K|^2 p(\theta_n) \Phi \tag{3.9}$$

Now, from (3.5) and assumption (3a) we get

$$\Phi_{hh} = \sum_{n=1}^{N} \Phi_{h_n h_n} = \left(\sum_{n=1}^{N} \cos^2 \theta_n \, p(\theta_n) \right) \Phi \tag{3.10}$$

Similarly from (3.8) and (3.9) we get

$$\Phi_{vh} = \left(\sum_{n=1}^{N} \cos \theta_n p(\theta_n) \right) K \Phi \tag{3.11}$$

and

$$\Phi_{vv} = \left(\sum_{n=1}^{N} p(\theta_n) \right) |K|^2 \Phi = |K|^2 \Phi \tag{3.12}$$

Therefore

$$C = \frac{|\Phi_{vh}|^2}{\Phi_{hh} \Phi_{vv}} = \frac{\left| \sum_{n=1}^{N} \cos \theta_n p(\theta_n) K \Phi \right|^2}{\left[\sum_{n=1}^{N} \cos^2 \theta_n p(\theta_n) \Phi \right] \left[|K|^2 \Phi \right]}$$

or, finally,

$$C = \frac{\left(\sum_{n=1}^{N} \cos \theta_n p(\theta_n) \right)^2}{\sum_{n=1}^{N} \cos^2 \theta_n p(\theta_n)} \tag{3.13}$$

We call $p(\theta_n)$ the (discrete) azimuthal power distribution function. Thus (3.13) gives the coherence C between vertical and horizontal for N discrete single-mode Rayleigh sources with azimuthal power distribution function $p(\theta_n)$.

B. CONTINUOUS SOURCE

It is of interest to derive a formula, similar to (3.13), giving the coherence for the case of uncorrelated single-mode Rayleigh noise arriving from the continuous range of azimuth $-\pi < \theta < \pi$, with a continuous azimuthal power density function $P(\theta)$.

The meaning of the underlined phrase requires some clarification. Throughout this study, noise arriving from a continuous range of azimuths will be regarded as a limiting case of the problem we have already considered involving a finite number N of uncorrelated sources, where $N \to \infty$ and the

III-4

azimuthal separation between sources $\to 0$. We will take the limit in such a way that the total vertical power remains constant.

Thus, suppose $P(\theta)$ is a non-negative, Riemann-integrable function defined on $\left|-\pi, \pi\right|$ such that

$$\int_{-\pi}^{\pi} P(\theta)\, d\theta = 1 \tag{3.14}$$

and suppose $\Phi(f)$ is an autopower spectrum such that

$$\int_{-\infty}^{\infty} \Phi(f)\, df = 1$$

We call $P(\theta)$ a <u>continuous azimuthal power distribution function</u>. Intuitively, we will think of $P(\theta)$ as defining the distribution of source power with regard to azimuth, for a single-mode Rayleigh source distributed over the whole range of possible azimuths.

Identifying the endpoints of the inteval $\left[-\pi, \pi\right]$, let $-\pi = \theta_N < \theta_1 < \theta_2 < \ldots < \theta_{N-1} < \theta_N = \pi$ and define

$$P(\theta_n) = \begin{cases} \displaystyle\int_{\theta_n}^{\theta_{n+1}} P(\theta)\, d\theta, & n = 1, 2, \ldots, N-1 \\[4mm] \displaystyle\int_{0}^{\theta_1} P(\theta)\, d\theta, & n = N \end{cases}$$

Then $\displaystyle\sum_{n=1}^{N} p(\theta_n) = 1$, and if we have uncorrelated Rayleigh sources at the azimuths $\theta_1, \ldots, \theta_N$ with inline horizontal autopower spectra $p(\theta_1)\Phi, \ldots, p(\theta_N)\Phi$, then the coherence will be, by (3.13),

$$C_{\theta_1, \ldots, \theta_n} = \frac{\left[\sum_{n=1}^{N} \cos \theta_n P(\theta_n)\right]^2}{\sum_{n=1}^{N} \cos^2 \theta_n P(\theta_n)} \qquad (3.16)$$

$$= \frac{\left[\sum_{n=1}^{N} \int_{\theta_n}^{\theta_{n+1}} \cos \theta_n P(\theta) \, d\theta\right]^2}{\sum_{n=1}^{N} \int_{\theta_n}^{\theta_{n+1}} \cos^2 \theta_n P(\theta) \, d\theta}$$

If we let N increase in such a way that $\min\limits_{1 \le n \le N} |\theta_{n+1} - \theta_n| \to 0$, then it is easily shown that

$$C_{\theta_1, \ldots, \theta_N} \to C = \frac{\left[\sum_{n=1}^{N} \int_{\theta_n}^{\theta_{n+1}} \cos \theta P(\theta) \, d\theta\right]^2}{\sum_{n=1}^{N} \int_{\theta_n}^{\theta_{n+1}} \cos^2 \theta P(\theta) \, d\theta}$$

$$= \frac{\left[\int_{-\pi}^{\pi} \cos \theta P(\theta) \, d\theta\right]^2}{\int_{-\pi}^{\pi} \cos^2 \theta P(\theta) \, d\theta}$$

In this sense we say that the formula

$$C = \frac{\left[\int_{-\pi}^{\pi} \cos \theta P(\theta) \, d\theta\right]^2}{\int_{-\pi}^{\pi} \cos^2 \theta P(\theta) \, d\theta} \qquad (3.17)$$

gives the coherence between vertical and horizontal for a two-component seismometer in a single-mode Rayleigh field with continuous azimuthal power distribution function $P(\theta)$.

C. CALCULATION OF COHERENCE FOR CERTAIN DISTRIBUTIONS $P(\theta)$

1. C = 1

Having obtained formulas (3.13) and (3.17), we shall now use these formulas to calculate coherence for several examples of azimuthal power distribution functions. The coherence C between v and h ranges from 0 (no prediction capability) to 1 (perfect predictability). A single discrete source will yield C=1, provided its azimuth θ is not equal to $\pm \frac{\pi}{2}$. More generally, (see formula (3.13)), C=1 whenever only two sources are present and their azimuths satisfy $\theta_1 = -\theta_2$, $\theta_1 \neq \pm \frac{\pi}{2}$. (Power may be distributed arbitrarily between these two sources; e.g., one source may have zero power.) It may be shown that this is the only situation for which C = 1.

2. C = 0

The other extreme, C=0, can occur in many ways. The most important cases for which C=0 are the following:

(1) Symmetry About $\frac{\pi}{2}$

Formula (3.17) shows that if $P(\frac{\pi}{2} - \theta) = P(\frac{\pi}{2} + \theta)$ for all θ, then C = 0.

(2) Periodicity

If there exists an integer $S \geq 2$ such that $P(\theta + \frac{2\pi}{S}) = P(\theta)$ for all θ, then C = 0. This follows from (3.7) and the fact that

$$\sum_{k=1}^{S} \cos\left(\frac{2\pi k}{S} + \delta\right) = 0$$

for all positive integers $S \geq 2$ and real numbers δ. An important example which satisfies both (1) and (2) is the isotropic distribution:

$$P(\theta) = \frac{1}{2\pi}$$

However, it is possible for the coherence to be quite large, even for certain rather widely distributed sources, as we shall illustrate with the three samples which follow. (See Figure III-1.)

3. Discrete Point Sources of Equal Power Located at Azimuths $\mu - \rho$ and $\mu + \rho$

Consider the discrete azimuthal power distribution function $p(\mu - \rho) = 1/2$, $p(\mu + \rho) = 1/2$. For this case, the coherence computed by (3.13) is

$$C_{\mu, \rho} = \frac{\left| \frac{1}{2} \cos(\mu - \rho) + \frac{1}{2} \cos(\mu + \rho) \right|^2}{\frac{1}{2} \cos^2(\mu - \rho) + \frac{1}{2} \cos^2(\mu + \rho)} = \frac{2 \cos^2 \mu \cos^2 \rho}{1 + \cos 2\mu \cos 2\rho} \tag{3.18}$$

A tabulation of the prediction error $1 - C_{\mu, \rho}$ for several values of μ and ρ is given in Table III-1.

4. Uniform Distribution From $\mu - \rho$ to $\mu + \rho$

Let
$$P(\theta) = \begin{cases} \frac{1}{2\rho}, & \mu - \rho \leq \theta \leq \mu + \rho \\ 0, & \text{otherwise} \end{cases}$$

This represents a uniform distribution of power coming from a wedge of azimuths with width 2ρ and center angle μ. By (3.17) the coherence is

$$C_{\mu, \rho} = \frac{\left[\frac{1}{2\rho} \int_{\mu - \rho}^{\mu + \rho} \cos \theta \, d\theta \right]^2}{\frac{1}{2\rho} \int_{\mu - \rho}^{\mu + \rho} \cos^2 \theta \, d\theta} = \frac{\left| \sin(\mu + \rho) - \sin(\mu - \rho) \right|^2}{2\rho \left[\rho + \frac{\sin(2\mu + 2\rho) - \sin(2\mu - 2\rho)}{4} \right]} \tag{3.19}$$

$$= \frac{4 \cos^2 \mu \sin^2 \rho}{\rho(2\rho + \cos 2\mu \sin 2\rho)}$$

A tabulation of $1 - C_{\mu, \rho}$ for certain values of μ and ρ is given in Table III-2.
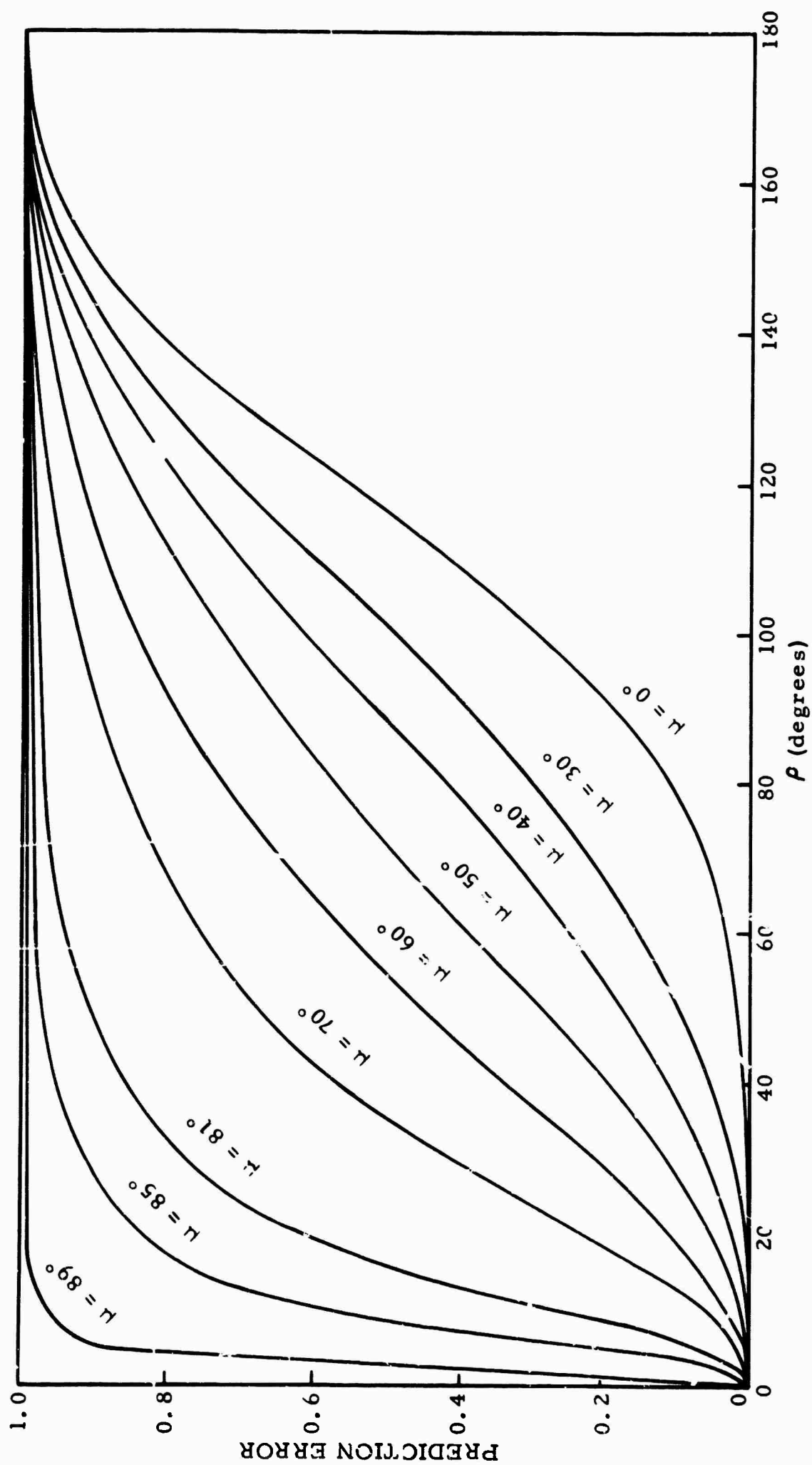
Figure III-1. Prediction Error for Horizontal and Vertical Instruments at Same Location when Power is Uniformly Distributed Between Azimuth $\mu - \zeta$ and $\mu + \zeta$

Table III-1

VALUES OF 1-C FOR TWO DISCRETE SOURCES OF EQUAL POWER
LOCATED AT AZIMUTHS, $\mu - \rho$ AND $\mu + \rho$. (μ, ρ IN DEGREES)

ρ \ μ	0	10	20	30	60	90
5 ;175	0.000	0.000	0.001	0.003	0.022	1.000
10 ;170	0.000	0.001	0.004	0.010	0.085	1.000
15 ;165	0.000	0.002	0.009	0.023	0.177	1.000
20 ;160	0.000	0.004	0.017	0.042	0.284	1.000
25 ;155	0.000	0.007	0.028	0.068	0.395	1.000
30 ;150	0.000	0.010	0.042	0.100	0.500	1.000
35 ;145	0.000	0.015	0.061	0.140	0.595	1.000
40 ;140	0.000	0.021	0.085	0.190	0.678	1.000
45 ;135	0.000	0.030	0.117	0.250	0.750	1.000
50 ;130	0.000	0.042	0.158	0.321	0.810	1.000
55 ;125	0.000	0.060	0.213	0.405	0.860	1.000
60 ;120	0.000	0.085	0.284	0.500	0.900	1.000
65 ;115	0.000	0.125	0.379	0.605	0.932	1.000
70 ;110	0.000	0.190	0.500	0.716	0.958	1.000
75 ;105	0.000	0.302	0.649	0.823	0.977	1.000
80 ;100	0.000	0.500	0.810	0.914	0.990	1.000
85 ;95	0.000	0.802	0.945	0.977	0.997	1.000
90		1.000	1.000	1.000	1.000	1.000

Table III-2

VALUES OF 1-C FOR UNIFORM DISTRIBUTION
FROM $\mu-\rho$ TO $\mu+\rho$ (μ, ρ IN DEGREES)

ρ \ μ	0	10	20	30	60	90
5	0.000	0.000	0.000	0.001	0.008	1.000
15	0.000	0.001	0.003	0.008	0.065	1.000
25	0.001	0.003	0.009	0.022	0.164	1.000
35	0.003	0.007	0.020	0.045	0.284	1.000
45	0.009	0.016	0.038	0.078	0.405	1.000
55	0.022	0.033	0.065	0.122	0.518	1.000
65	0.046	0.060	0.104	0.181	0.616	1.000
75	0.086	0.105	0.161	0.254	0.699	1.000
85	0.148	0.171	0.238	0.343	0.768	1.000
95	0.238	0.264	0.336	0.444	0.824	1.000
105	0.357	0.382	0.452	0.553	0.870	1.000
115	0.496	0.518	0.578	0.662	0.907	1.000
125	0.641	0.657	0.702	0.763	0.936	1.000
135	0.771	0.782	0.810	0.849	0.959	1.000
145	0.874	0.879	0.894	0.915	0.976	1.000
155	0.943	0.945	0.952	0.961	0.989	1.000
165	0.982	0.983	0.985	0.987	0.996	1.000
175	0.998	0.998	0.999	0.999	1.000	1.000
180	1.000	1.000	1.000	1.000	1.000	1.000

5. $\cos^2$ Distribution

Consider the continuous azimuthal power density function

$$P(\theta) = \begin{cases} \dfrac{1}{\rho} \cos^2\left(\dfrac{\pi}{2\rho}(\theta - \mu)\right), & \mu - \rho \leq \theta \leq \mu + \rho \\ 0, & \text{otherwise} \end{cases}$$

Using formula (3.17), we find, for $\rho \neq \dfrac{\pi}{2}$, $\rho \neq \pi$

$$C_{\mu,\rho} = \frac{\left(\dfrac{2\pi^2}{\pi^2 - \rho^2}\right)^2 \cos^2\mu \sin^2\rho}{2\rho\left[\rho + \dfrac{\pi^2}{2\pi^2 - 8\rho^2}\cos 2\mu \sin 2\rho\right]}. \tag{3.20}$$

and

$$C_{\mu,\frac{\pi}{2}} = \left(\frac{16}{3\pi}\right)^2 \frac{\cos^2\mu}{1 + 2\cos^2\mu} \tag{3.21}$$

$$C_{\mu,\pi} = \frac{\cos^2\mu}{2}$$

A tabulation of $1 - C_{\mu,\rho}$ for several values of μ, ρ is given in Table III-3.

Table III-3

VALUES OF 1-C FOR $\cos^2$ DISTRIBUTION
(μ, ρ IN DEGREES)

ρ \ μ	0	10	20	30	60	90
5	0.000	0.000	0.000	0.000	0.003	1.000
15	0.000	0.000	0.001	0.003	0.026	1.000
25	0.000	0.001	0.004	0.008	0.070	1.000
35	0.001	0.002	0.007	0.017	0.129	1.000
45	0.002	0.005	0.013	0.029	0.199	1.000
55	0.005	0.009	0.021	0.044	0.273	1.000
65	0.010	0.016	0.032	0.064	0.348	1.000
75	0.019	0.026	0.048	0.089	0.420	1.000
85	0.031	0.040	0.068	0.118	0.488	1.000
90	0.039	0.049	0.080	0.135	0.520	1.000
95	0.049	0.060	0.094	0.154	0.550	1.000
105	0.073	0.086	0.126	0.195	0.607	1.000
115	0.106	0.120	0.165	0.241	0.657	1.000
125	0.146	0.162	0.211	0.293	0.703	1.000
135	0.196	0.213	0.265	0.349	0.743	1.000
145	0.253	0.271	0.324	0.409	0.779	1.000
155	0.319	0.336	0.389	0.471	0.811	1.000
165	0.389	0.406	0.456	0.534	0.839	1.000
175	0.463	0.479	0.525	0.595	0.864	1.000
180	0.500	0.515	0.558	0.625	0.875	1.000

BLANK PAGE

SECTION IV
EFFECT OF ADDITIONAL RANDOM UNCORRELATED NOISE

The coherence between vertical and horizontal for the case we have considered in Section III will be reduced if the seismometer outputs v and/or h contain an additional component of random, uncorrelated noise; e.g., amplifier noise.

Consider the following quite general problem: we are given two stationary random processes v and h, with coherence C, if $v^1 = v + z$, $h^1 = h + x$, where z and x are stationary random processes uncorrelated to v, h, and uncorrelated to each other, then what is the coherence C^1 between v^1 and h^1?

We have

$$\Phi_{v^1 h^1} = \Phi_{vh} \tag{4.1}$$

$$\Phi_{v^1 v^1} = \Phi_{vv} + \Phi_{zz}$$

$$\Phi_{h^1 h^1} = \Phi_{hh} + \Phi_{xx}$$

Therefore,

$$C^1 = \frac{|\Phi_{vh}|^2}{\left(\Phi_{vv} + \Phi_{zz}\right)\left(\Phi_{hh} + \Phi_{xx}\right)} = \frac{1}{\left(1 + \dfrac{\Phi_{zz}}{\Phi_{vv}}\right)} \cdot \frac{1}{\left(1 + \dfrac{\Phi_{xx}}{\Phi_{hh}}\right)} \cdot C \tag{4.2}$$

We see from equation (4.2) that the effect on coherence depends upon the ratios Φ_{zz}/Φ_{vv} and Φ_{xx}/Φ_{hh}. In the case that v and h are the vertical or horizontal components of displacement due to a single-mode Rayleigh field with some azimuthal power distribution, $P(\theta)$, and if we conceive that Φ_{zz}, Φ_{vv}, and Φ_{xx} remain fixed as we vary $P(\theta)$, we see that the

greatest reduction in coherence results when $P(\theta)$ is strongly concentrated near $\pm\dfrac{\pi}{2}$, for then, by equation (3.10), Φ_{hh} will be very small. For example, if $\Phi_{hh} < \Phi_{xx}$, then $C^1 < 1/2\, C$, by (4.2).

For any real numbers α, β with $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, it is true that

$$\left(\frac{1}{1+\alpha}\right)\left(\frac{1}{1+\beta}\right) \geq 1 - \alpha - \beta$$

Therefore, if

$$\Phi_{zz} \leq \Phi_{vv}, \quad \Phi_{xx} \leq \Phi_{hh} \tag{4.3}$$

then

$$C^1 \geq \left[1 - \frac{\Phi_{zz}}{\Phi_{vv}} - \frac{\Phi_{xx}}{\Phi_{hh}}\right] C \tag{4.4}$$

Formula (4.3) is a reasonable assumption if Φ_{zz} and Φ_{xx} are interpreted as amplifier noise, and $P(\theta)$ is not concentrated near $\pm\dfrac{\pi}{2}$. Thus, for example, (4.3) shows that if on each channel the power of amplifier noise is less than 1 percent of the power of seismic noise, then the coherence is reduced by less than 2 percent.

We may conclude that for the case of single-mode Rayleigh waves with an azimuthal power density $P(\theta)$, the coherence between v and h will not be seriously affected by the addition of a realistic amount of amplifier noise on both channels, unless $P(\theta)$ is concentrated near $\pm\dfrac{\pi}{2}$.

SECTION V
LOVE WAVES

We shall discuss in this section the limitation upon prediction capability imposed by the addition of Love (SH) waves on the horizontal channel. It will always be assumed that the Love waves are uncorrelated with the Rayleigh waves.

A. GENERAL FORMULA FOR EFFECT OF LOVE WAVES ON COHERENCE

Suppose we have N sources, at azimuths $\theta_1, \ldots, \theta_N$, each source generating Rayleigh noise, with components v_n, λ_n, h_n as in Section III. Suppose also that the nth source generates Love wave noise with motion j_n in the direction $\theta_n + \pi/2$.

In addition to the assumptions of Section III, paragraph A, we assume

(1)   For each n, $j_n(t)$ is a stationary random process

(2)   There exists an autopower spectrum $\Omega(f)$ and constants $q(\theta_n)$ such that $\sum_{n=1}^{N} q(\theta_n) = 1$ and $\Phi_{j_n j_n} = q(\theta_n) \Omega$, $n = 1, 2, \ldots, N$.

and

(3)   The crosspower spectra $\Phi_{j_m j_n} = 0$, $m \neq n$ and
The crosspower spectra $\Phi_{j_m \lambda_n} = 0$ for all m, n.

Let us define

$$w_n(t) = \sin \theta_n j_n(t) \tag{5.1}$$

and

$$w(t) = \sum_{n=1}^{N} w_n(t) \tag{5.2}$$

Clearly w_n is the contribution to the output of the horizontal seismometer due to Love waves from the nth source, and $w(t)$ is the total Love component on the horizontal.

Let $L(f)$ be defined by the relationship

$$\Omega(f) = L(f) \, \Phi(f) \tag{5.3}$$

Then (cf. 3.10) we find

$$\Phi_{ww} = \sum_{n=1}^{N} \sin^2 \theta_n \, q(\theta_n) \, \Omega \tag{5.4}$$

$$= \sum_{n=1}^{N} \sin^2 \theta_n \, q(\theta_n) \cdot L \cdot \Phi$$

Thus, if C_L = coherence between v and $h+w$,

$$C_L = \frac{|\Phi_{vh}|^2}{|\Phi_{hh} + \Phi_{ww}| \, \Phi_{vv}} \tag{5.5}$$

$$= \frac{\left[\sum_{n=1}^{N} \cos \theta_n \, p(\theta_n)\right]^2 \, |K|^2 \, \Phi^2}{\left[\sum_{n=1}^{N} \cos^2 \theta_n \, p(\theta_n) \, \Phi + L \sum_{n=1}^{N} \sin^2 \theta_n q(\theta_n) \, \Phi\right] |K|^2 \Phi}$$

by Equations (3.10) - (3.12), (5.4). Hence

$$C_L = \frac{\left[\sum_{n=1}^{N} \cos \theta_n p(\theta_n)\right]^2}{\sum_{n=1}^{N} \cos^2 \theta_n \, p(\theta_n) + L \sum_{n=1}^{N} \sin^2 \theta_n q(\theta_n)}$$

We call $q(\theta_n)$ the (discrete) azimuthal power distribution function for the Love waves. As before, we can obtain a formula similar to (5.6) giving the coherence for a continuous azimuthal power distribution $Q(\theta)$ for the Love waves:

$$C_L = \frac{\left| \int_{-\pi}^{\pi} \cos \theta \, P(\theta) \, d\theta \right|^2}{\int_{-\pi}^{\pi} \cos^2 \theta \, P(\theta) \, d\theta + L \cdot \int_{-\pi}^{\pi} \sin^2 \theta \, Q(\theta) \, d\theta} \qquad (5.7)$$

Note that (5.6) and (5.7) reduce to (3.13) and (3.17) respectively, on setting $L = 0$.

B. ISOTROPIC LOVE WAVES

It is best to regard the case of isotropic Love waves $\left(Q(\theta) = \frac{1}{\pi} \right)$ as a special case of the situation discussed in Section IV. That is, the Love wave energy on the horizontal is simply uncorrelated additive noise on the predicting channel, and its effect upon coherence is mathematically indistinguishable from that of, say, amplifier noise.

Thus, if C is the coherence between v and h, we have by (4.2) that

$$C_L = \left[\frac{\Phi_{hh}}{\Phi_{hh} + \Phi_{ww}} \right] \cdot C \qquad (5.8)$$

If Φ_{ww} is large, there is a substantial reduction in coherence. For, recall (cf. 3.10) that

$$\Phi_{hh} = \left[\int_{-\pi}^{\pi} \cos^2 \theta \, P(\theta) \, d\theta \right] \cdot \Phi \leq \Phi$$

Also, by (5.4)

$$\Phi_{ww} = \left[\int_{-\pi}^{\pi} \sin^2 \theta \cdot \frac{1}{2\pi} \, d\theta \right] \cdot L \cdot \Phi = \frac{1}{2} L \Phi$$

Therefore

$$\frac{\Phi_{hh}}{\Phi_{hh} + \Phi_{ww}} \leq \frac{\Phi}{\Phi + \frac{1}{2} L \Phi} = \frac{2}{2 + L}$$

Hence,

$$C_L \leq \left(\frac{2}{2+L}\right) \cdot C \qquad \left|\text{isotropic Love waves}\right| \qquad (5.9)$$

Formula (5.9) implies, for example, that if $L = 2$, then the coherence C_L is never greater than $1/2$.

C. LOVE WAVES WITH SAME DISTRIBUTION AS THE RAYLEIGH WAVES

If the Love wave field is non-isotropic, formula 5.8 still holds. But in this case it may be misleading to regard the Love waves simply as additive uncorrelated noise, because ϕ_{ww} depends upon the orientation of our horizontal seismometer. That is, by pointing our horizontal seismometer in the proper direction, it may be possible to make ϕ_{ww} quite small.

To illustrate this phenomenon, consider the situation in which the Love waves have the same azimuthal power distribution as the Rayleigh waves; i.e.,

$$Q(\theta) = P(\theta)$$

Then (5.7) becomes

$$C_L = \frac{\left[\int_{-\pi}^{\pi} \cos\theta\, P(\theta)\, d\theta\right]^2}{\int_{-\pi}^{\pi} \cos^2\theta\, P(\theta)\, d\theta + L \int_{-\pi}^{\pi} \sin^2\theta\, P(\theta)\, d\theta} \qquad (5.11)$$

If in the distributions $P(\theta)$ considered in Section III, paragraph C, we specify the values of ρ and L, then the coherence C_L calculated by (5.11) is maximized for $\mu = 0$. For $\mu = 0$, we get surprisingly high coherence, even for L as large as 10 (See Tables V-1, V-2). For example, for a uniform distribution from $-25°$ to $+25°$ and $L = 10$, the prediction error is only 0.395 (Table V-2).

Table V-1

VALUES OF 1-C FOR DISCRETE SOURCES AT AZIMUTHS ± ρ
WITH INLINE LOVE POWER = L x INLINE RAYLEIGH POWER

ρ (Degrees)	L = 0	L = 0.1	L = 1.0	L = 10.0
5; 175	0.000	0.001	0.008	0.071
15; 165	0.000	0.007	0.067	0.418
25; 155	0.000	0.021	0.179	0.685
35; 145	0.000	0.047	0.329	0.831
45; 135	0.000	0.091	0.500	0.909
55; 125	0.000	0.169	0.671	0.953
65; 115	0.000	0.315	0.821	0.979
75; 105	0.000	0.582	0.933	0.993
85; 95	0.000	0.929	0.992	0.999

Table V-2

VALUES OF 1-C FOR UNIFORM RAYLEIGH DISTRIBUTION FROM -ρ TO +ρ
WITH INLINE LOVE POWER = L x INLINE RAYLEIGH POWER

ρ (Degrees)	L = 0	L = 0.1	L = 1.0	L = 10.0
5	0.000	0.000	0.006	0.025
15	0.000	0.002	0.023	0.187
25	0.001	0.007	.064	0.395
35	0.003	0.016	0.118	0.568
45	0.009	0.030	0.189	0.692
55	0.022	0.054	0.271	0.779
65	0.046	0.091	0.361	0.840
75	0.086	0.143	0.455	0.882
85	0.148	0.217	0.549	0.914
95	0.238	0.314	0.639	0.937
105	0.357	0.431	0.722	0.955
115	0.496	0.561	0.796	0.968
125	0.641	0.689	0.859	0.978
135	0.771	0.802	0.910	0.986
145	0.874	0.890	0.949	0.992
155	0.943	0.950	0.976	0.996
165	0.982	0.984	0.992	0.999
175	0.998	0.998	0.999	1.000
180	1.000	1.000	1.000	1.000

BLANK PAGE

SECTION VI

EFFECT OF GAIN FLUCTUATION

In the preceding sections we have assumed implicitly that the amplitude gains of our two seismometers are constant functions of time. In practice, however, these gains vary slowly with time, and are never known precisely. This results in a lower observable coherence between the two channels than is indicated by our previous equations. Fortunately, this effect is not serious if the gain fluctuations are reasonably small and well-behaved.

For, let $r = r(t)$ and $s = s(t)$ be the gains of the vertical and horizontal seismometers respectively. We assume that r and s are non-negative variables satisfying:

(1) $r(t)$ and $s(t)$ are statistically independent stationary random processes, with first order probability functions Q_r and Q_s, respectively

(2) During any short interval of time during which we continuously record data for use in an experimental determination of φ_{vv}, φ_{vh}, or φ_{hh}, the variations in the gains are so slight that we may consider the gains to be constant throughout that interval

With these assumptions, suppose now that a certain noise situation persists for many days; that is, the actual correlations φ_{vv}, φ_{vh}, and φ_{hh}, and the azimuthal power distribution function $P(\vartheta)$ remain unchanged over an extended period of time. During this period let us repeatedly use our two seismometers to measure φ_{vv}, φ_{vh}, and φ_{hh}. The gains r and s are varying during this period, but in accordance with assumption (2), we make the approximation that r and s remain constant

throughout each interval of measurement. Then, the observed correlations determined by measurement beginning at time t are

$$r(t)^2 \, \varphi_{vv}, \quad r(t) \, s(t) \, \varphi_{vh}, \quad \text{and} \quad s(t)^2 \, \varphi_{hh}$$

Taking the averages of these correlations for many different values of t, we obtain the _average observed correlations_ ϕ_{vv}, ϕ_{vh}, and ϕ_{hh}. Thus

$$\phi_{vv} = \overline{r(t)^2} \; \varphi_{vv}$$

$$\phi_{vh} = \overline{r(t) \, s(t)} \; \varphi_{vh} \tag{6.1}$$

$$\phi_{hh} = \overline{s(t)^2} \; \varphi_{hh}$$

We define C'', the _observed coherence_, by

$$C'' = \frac{\left| \phi''_{vh} \right|^2}{\phi''_{vv} \, \phi''_{hh}} = \frac{\left(\overline{r(t) \, s(t)} \right)^2 \left| \phi_{vh} \right|^2}{\overline{r(t)^2} \; \overline{s(t)^2} \; \phi_{vv} \, \phi_{hh}} \tag{6.2}$$

Therefore,

$$C'' = \frac{\left(\overline{r(t) \, s(t)} \right)^2}{\overline{r(t)^2} \; \overline{s(t)^2}} \; C \tag{6.3}$$

where C is the coherence between v and h that would be observed by seismometers having no gain fluctuation.

The mathematical significance of C'' lies in the fact that $1 - C''$ is the relative prediction error corresponding to the optimum Wiener prediction filter transfer function G(f) to be applied to h for predicting v in the variable gain situation.

In greater detail: let $g(t)$ be a convolution operator, $G(f) =$ Fourier transform of g, and define $e = v - (g \circledS h)$. Let ϕ_{ee} be the average observed autocorrelation function of e (cf. the definition of ϕ_{vv}, ϕ_{vh} and ϕ_{hh}), and let $P_e(f)$ be the Fourier transform of ϕ_{ee}.

Take $G(f) = \dfrac{A(f)}{B(f)}$, where A and B are the Fourier transforms of ϕ_{vh} and ϕ_{hh} respectively; it may be shown that this choice of the filter transfer function G is optimal in the sense that it minimizes the average error power

$$\int_{-\infty}^{\infty} P_e(f)\, df$$

Furthermore, we find that for this choice of G,

$$\int_{-\infty}^{\infty} P_e(f)\, df = D(f)\left[1 - C''(f)\right]$$

where $D(f)$ is the Fourier transform of ϕ_{vv}.

Now, by ergodicity of the time series $r(t)$ and $s(t)$, (6.3) may be written

$$C'' = \frac{\left[\int_0^{\infty}\int_0^{\infty} Q_r(x)\, Q_s(y)\, dx\, dy\right]^2}{\int_0^{\infty} Q_r(x)^2\, dx \int_0^{\infty} Q_s(y)^2\, dy} \cdot C \qquad (6.4)$$

or, since $r(t)$ and $s(t)$ are statistically independent,

$$C'' = \frac{\left[\int_0^{\infty} Q_r(x)\, dx\right]^2 \left[\int_0^{\infty} Q_s(y)\, dy\right]^2}{\int_0^{\infty} Q_r(x)^2\, dx \cdot \int_0^{\infty} Q_s(y)^2\, dy} \cdot C \qquad (6.5)$$

$$= \left(\frac{\mu_r^2}{\mu_r^2 + \sigma_r^2}\right)\left(\frac{\mu_s^2}{\mu_s^2 + \sigma_s^2}\right) C$$

where μ_r and μ_s are the means of the distributions Q_r and Q_s respectively, and σ_r^2 and σ_s^2 are the corresponding variances.

It is clear from formula (6.5) that the effect on coherence due to gain fluctuation is small if the variances σ_r^2 and σ_s^2 are small compared to μ_r^2 and μ_s^2. To illustrate the quantitative effect of gain fluctuation on coherence, we consider two examples of gain probability distributions.

Example 1. Uniform Distribution from 0 to R

Let R be an arbitrarily large positive number and let

$$Q_r(x) = Q_s(x) = \begin{cases} \dfrac{1}{R}, & 0 \le x \le R \\ 0, & \text{otherwise} \end{cases}$$

Then compute from (6.5) that

$$C'' = \frac{9}{16} C$$

Thus in the rather extreme case that the gains are evenly distributed between 0 and some large number, the coherence is reduced by less than a factor of 2.

Example 2. Log (gain) Normally Distributed

Suppose that the random variables log r and log s both are normally distributed with mean 0 and variance σ^2. Then

$$\mu_r = \mu_s = \int_0^\infty x\, Q_r(x)\, dx = \int_{-\infty}^\infty e^y \left[\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-y^2}{2\sigma^2} \right) \right] dy$$

$$= e^{\frac{\sigma^2}{2}} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^\infty \exp\left(\frac{-\left(y - \sigma^2\right)^2}{2\sigma^2} \right) dy = e^{\frac{\sigma^2}{2}} \quad \sigma^2/2$$

Also,

$$\mu_r^2 + \sigma_r^2 = \mu_s^2 + \sigma_s^2 = \int_0^\infty x^2 Q_r(x)\, dx = \int_{-\infty}^\infty e^{2y} \left[\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-y^2}{2\sigma^2} \right) \right] dy$$

$$= e^{2\sigma^2} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^\infty \exp\left(\frac{-\left(y - 2\sigma^2\right)^2}{2\sigma^2} \right) dy = e^{2\sigma^2}$$

Therefore, by (6.5)

$$C'' = \frac{\left(e^{\frac{\sigma^2}{2}}\right)^2}{e^{2\sigma^2}} \cdot \frac{\left(e^{\frac{\sigma^2}{2}}\right)^2}{e^{2\sigma^2}} \cdot C = e^{-2\sigma^2} C \qquad (6.6)$$

In Table VI-1, μ_r, σ_r^2, and $e^{-2\sigma^2}$ are tabulated for several values of σ^2.

We see from Table VI-1 that in Example 2 (which is probably a more realistic gain distribution than Example 1) a severe loss in coherence results if σ_r^2 is allowed to be as large as, say, 1. It should be possible, however, through a program of peridical calibration, to insure that the seismometer gain fluctuations remain within acceptable limits. This will be necessary if one is to implement a processor in which the prediction filter applied to h for predicting v remains unchanged for an extended period of time.

Table VI-1

EFFECT OF GAIN FLUCTUATION ON COHERENCE FOR log (gain)

NORMALLY DISTRIBUTED, MEAN = 0, VARIANCE = σ^2

σ^2	μ_r	σ_r^2	$e^{-2\sigma^2}$
0	1	0	1
0.01	1.0051	0.0101	0.9802
0.1	1.0513	0.1162	0.8187
0.3	1.1618	0.4722	0.5488
0.5	1.2840	1.0696	0.3679
1.0	1.6487	4.6708	0.1353

BLANK PAGE

SECTION VII

MULTIMODE NOISE

In Sections III through VI we have discussed the coherence between vertical and horizontal seismometers at point location in a single-mode noise field, with the possible addition of Love waves on the horizontal. For the same two-channel system, we shall now obtain formulas for coherence between v and h in the more general case that not one, but several modes of noise are present.

As in Section III, assume we have a two-component seismometer located at the origin O of an X-Y coordinate system in the plane. Let $\theta_1, \theta_2, \ldots, \theta_N$ be azimuths; for each $n = 1, 2, \ldots, N$, we have a noise generator at azimuth θ_n.

Let v_n and λ_n be the respective outputs, due to noise from the nth source, of a vertical seismometer at P and an inline horizontal seismometer at P.

We assume that [*]

$$v_n = \sum_{m=0}^{M} v_n^m , \quad \lambda_n = \sum_{m=0}^{M} \lambda_n^m \qquad (7.1)$$

where v_n^m and λ_n^m are respectively the vertical and inline horizontal components of mth noise mode from the nth source, $m = 0, 1, 2, \ldots, M$.

Also, we assume that the nth source generates Love (SH) noise with motion j_n in the direction $\theta_n + \dfrac{\pi}{2}$.

We make the following basic assumptions about v_n^m, λ_n^m, and j_n:

---

[*]

Notation: numerical subscripts refer to the source; superscripts refer to the mode.

(1) λ_n^m and j_n are stationary random processes, for all n, m.

(2)   For each m = 0, 1, 2, . . ., M, there exists an autopower spectrum ϕ^m, and constants $p^m(\theta_n)$, such that

$$\sum_{n=1}^{N} p^m(\theta_n) = 1$$

and

$$\phi_{\lambda_n^m \lambda_n^m} = p^m(\theta_n) \, \phi^m, \quad n = 1, 2, \ldots, N$$

There exists an autopower spectrum Ω and constants $q(\theta_n)$ such that

$$\sum_{n=1}^{N} q(\theta_n) = 1$$

and

$$\phi_{j_n j_n} = q(\theta_n) \, \Omega, \quad n = 1, 2, \ldots, N$$

(3) λ_m^n and λ_μ^ν are uncorrelated unless $n = \nu$ and $m = \mu$.

λ_m^n and j_μ are uncorrelated for all n, m, μ.

(4)   For each m, there exists a function $k^m(t)$ with

$$\int_{-\infty}^{\infty} |k^m(t)| \, dt < \infty$$

such that

$$v_n^m = k^m \circledast \lambda_n^m \quad \text{for all } n = 1, 2, \ldots, N$$

Having made these assumptions, let us define

$$h_n(t) = -\cos \vartheta_n \, \lambda_n(t) \tag{7.2}$$

$$w_n(t) = \sin \vartheta_n \, j_n(t)$$

$$h(t) = \sum_{n=1}^{N} h_n(t)$$

$$w(t) = \sum_{n=1}^{N} w_n(t)$$

$$v(t) = \sum_{n=1}^{N} v_n(t)$$

Clearly $h_n + w_n$ is the contribution to the output of our horizontal seismometer due to the nth source (w_n is the Love wave contribution); $h(t) + w(t)$ and $v(t)$ are respectively, the total horizontal and vertical seismograms.

We proceed to find the coherence

$$C = \frac{\left| \Phi_{v,\,h+w} \right|^2}{\Phi_{vv} \, \Phi_{h+w,\,h+w}}$$

between v and $h+w$.

As before, let K^m be the Fourier transform of k^m, $m = 0$, $1, 2, \ldots, M$, and let $L(f)$ be defined by

$$\Omega(f) = L(f) \sum_{m=0}^{M} \Phi^m(f) \tag{7.4}$$

$\left| L(f) \text{ is thus a measure of the ratio of inline Love power to total inline SV+P power .} \right|$

Omitting the calculations, which are but a slight generalization of the corresponding calculations in Sections III and V, we find

$$\Phi_{v, h+w} = \sum_{n=1}^{N} \sum_{m=0}^{M} \cos \theta_n \, p^m(\theta_n) \, K^m \, \Phi^m \tag{7.5}$$

$$\Phi_{vv} = \sum_{m=0}^{M} \left| K^m \right|^2 \Phi^m$$

and

$$\Phi_{h+w, \, h+w} = \sum_{n=1}^{N} \sum_{m=0}^{M} \cos^2 \theta_n \, p^m (\theta_n) \, \Phi^m$$

$$+ \sum_{n=1}^{N} \sin^2 \theta_n \, q(\theta_n) \, \Omega$$

One can now substitute (7.5) into (7.3) and obtain the general multimode coherence. However, we shall now restrict ourselves to a simpler case; namely, we make one additional assumption:

(5)   All of the azimuthal power distribution functions are equal, i.e., $p^0 = p^1 = \ldots = p^m = q$.

Letting p be the azimuthal power distribution function common to all modes, in accordance with assumption (5), we find from (7.3), (7.4) and (7.5) that the coherence is:

$$\tag{7.6}$$

$$C \; = \; \frac{\left(\sum_{n=1}^{N} \cos \theta_n p(\theta_n) \right)^2}{\sum_{n=1}^{N} \cos^2 \theta_n p(\theta_n) + L \sum_{n=1}^{N} \sin^2 \theta_n p(\theta_n)} \; \cdot \; \frac{\left| \sum_{m=0}^{M} K^m \Phi^m \right|^2}{\sum_{m=0}^{M} \Phi^m \cdot \sum_{m=0}^{M} \left| K^m \right|^2 \Phi^m}$$

As before, we can write a similar formula for the case of a continuous azimuthal power distribution function $P(\theta)$;

(7.7)

$$C = \frac{\left[\int_{-\pi}^{\pi} \cos\theta\, P(\theta)\, d\theta\right]^2}{\int_{-\pi}^{\pi}\cos^2\theta\, P(\theta)\, d\theta + L\int_{-\pi}^{\pi}\sin^2\theta\, P(\theta)\, d\theta} \cdot \frac{\left|\sum_{m=0}^{M} K^m \Phi^m\right|^2}{\sum_{m=0}^{M}\Phi^m \cdot \sum_{m=0}^{M}\left|K^m\right|^2 \Phi^m}$$

Formula for multimode coherence, all modes having same azimuthal power distribution function $P(\theta)$.

Comparison of (7.7) with (5.11) shows that (Multimode coherence) = Q (f) X (Single-mode coherence), where

$$Q = \frac{\left|\sum_{m=0}^{M} K^m \Phi^m\right|^2}{\sum_{m=0}^{M}\Phi^m \cdot \sum_{m=0}^{M}\left|K^m\right|^2 \Phi^m} \qquad (7.8)$$

is a function depending only on the autopower spectra Φ^m and modal horizontal-vertical transfer functions K^m. Since we have already investigated single-mode coherence in Sections III through VI, the study of multimode coherence for the case that all modes have the same azimuthal power distribution function reduces to a consideration of the function $Q(f)$.

Detailed examination of the behavior of $Q(f)$ for realistic choices of the Φ^m's and K^m's has not been carried out. However, we can make a few elementary remarks.

First, it should be noted that the presence of more than one mode always results in a reduction in coherence from the single-mode case. For, it follows from Hölder's inequality that (for real frequencies) $0 \leq Q(f) \leq 1$. Furthermore, assuming all Φ^m to be non-zero, we have $Q(f) < 1$ unless $K^0 = K^1 = \ldots = K^m$.

In a physical situation, it is extremely unlikely that all the transfer functions K^m would be equal, or nearly so. Approximately, we might expect that there would exist positive constants $R_0, R_1, \ldots, R_M$ such that over a fairly broad band of frequencies we have

$$K^m = \pm iR_m, \quad m = 0, 1, \ldots, M$$

Take as an example the case $M = 1$ (two modes), and let $\Phi^0 = \Phi^1$. Let R be any positive number. Table VII-1 shows values of Q computed by (7.8) for several choices of K^0, K^1.

Table VII-1

VALUES OF Q FOR TWO-MODE CASE, $\Phi^0 = \Phi^1$

K_0	K_1	Q
+ i R	+ i R	1
+ i R	+ 2 i R	9/10
+ i R	+ 5 i R	9/13
+ i R	+ b i R	$(b+1)^2 / 2(b^2 + 1)$, b > 0
+ i R	- i R	0

Table VII-1 illustrates a general principle, which is roughly as follows: the reduction in coherence due to additional modes is most severe in the case that some of the K^m's represent $+90°$ phase shift and others, a $-90°$ phase shift; for frequencies where all K^m's have the same phase response, the effect is less marked. Indeed, for the two-mode case, we get as low as $Q = 0$ if K^0 and K^1 have opposite phase responses, whereas $Q > \frac{1}{2}$ if K^0 and K^1 have the same phase responses.

Given an azimuthal power distribution function $P(\theta)$ it is apparent that if several modes are present with approximately equal power in each mode, then the coherence is likely to be substantially reduced from that for a single mode with the same distribution $P(\theta)$. On the other hand, in closing this section it should be remarked that it also follows from (7.8) that if one mode--say, the fundamental Rayleigh mode--accounts for all but a small amount of the total power, then there will be very little reduction in coherence from the single-mode case.

BLANK PAGE

SECTION VIII
DISCUSSION

In the preceding sections, mathematical formulae have been derived for the noise-rejection properties of a 2-component seismometer, consisting of 1 vertical-component and 1 horizontal-component instrument at the same point. For most applications, this system may be considered equivalent to a 3-component seismometer. It is possible to design a processing system wherein different filters are applied to the outputs of 2 horizontals and the noise estimate is obtained by summing the filter outputs. This case is not covered by the development of Part I, but the simpler case, in which the horizontal outpus are summed and then a filter is applied, has been treated explicitly. This is because an arbitrarily weighted sum of 2 horizontal components of ground motion must always be equivalent to a single component in a direction defined by the weights.

Some general conclusions may be stated regarding the usefulness of a single horizontal-component instrument for eliminating noise from the output of a vertical instrument at the same location. It would appear that such effects as system noise, gain fluctuations and Love waves need not present serious problems, although in some cases they may very well do so. The most important consideration is usually the properties of the noise to be eliminated. If significant noise rejection is to be achieved, then some rather stringent conditions must be satisfied. Even when there is only a single-noise mode present, it is essential that the azimuths of the noise sources be confined within a range somewhat less than 180° and that the horizontal be oriented near the center of this range. This requirement follows from the obvious fact that a filter which cancels out noise traveling in a given direction must also amplify noise traveling in the opposite direction. In practice, such extreme directional properties are seldom found.

If more than one noise mode is important, large differences in the horizontal-to-vertical transfer functions, k(t), cannot be permitted. In Section VII, a two-mode situation is presented in which no noise rejection is possible because one mode is characterized by a prograde particle motion and the other is associated with retrograde motion.

With a 3-component seismometer at point location, it is theoretically possible to determine both the direction of propagation and the shape of the particle orbit for an observed Rayleigh wave. However, the sense of propagation and the sense of the orbital motion cannot be found unless a spatial separation is introduced into the system. For this reason it is concluded that ordinary 3-component systems offer little promise for most applications and that attention must be directed toward arrays of horizontal- and vertical-component seismometers.

PART II

MULTICOMPONENT ARRAYS

BLANK PAGE

SECTION IX

FORMULAS FOR ARRAY RESPONSE IN SINGLE-MODE NOISE WITH ARBITRARY AZIMUTHAL POWER DISTRIBUTION

A. DESCRIPTION OF ARRAY

Let O be the origin of an X-Y coordinate system in the plane of the earth's surface. We consider an array of seismometers consisting of a vertical seismometer v located at O, and M horizontal seismometers h_1, h_2, . . . , h_M, where the mth horizontal seismometer h_m is located at the point P_m having polar coordinates (r_m, ρ_m), and h_m is oriented in the direction ψ_m (Figure IX-1).
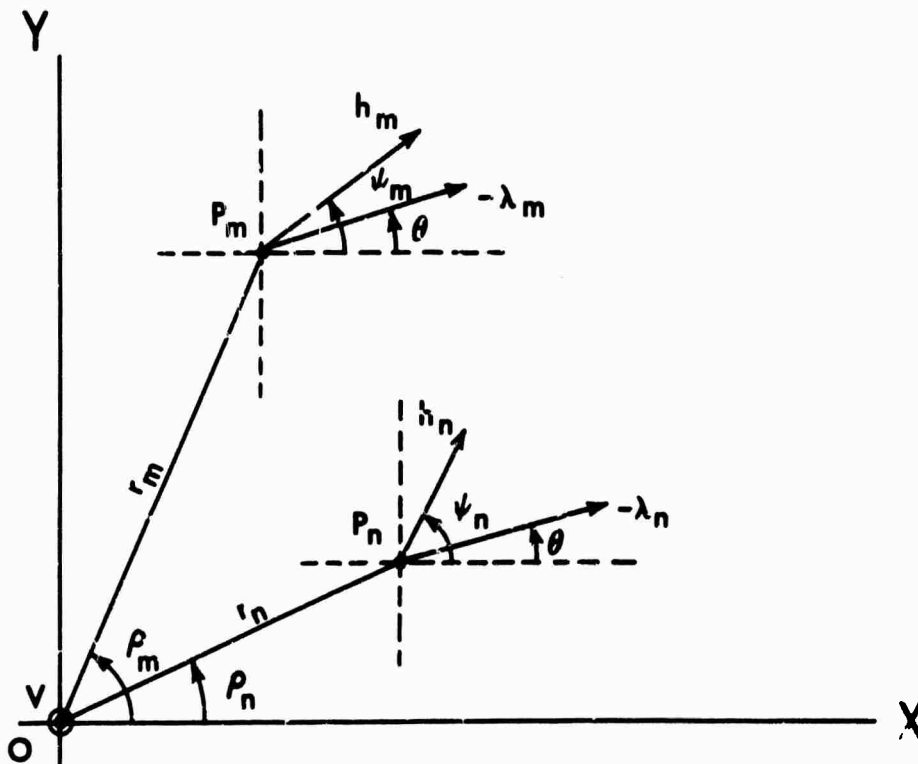

Figure IX-1. Generalized Seismometer Array Configuration

B. ASSUMPTIONS ABOUT THE NOISE FIELD

We wish to examine, for this array, the extent to which noise on the vertical can be predicted from noise on the horizontals in single-mode trapped noise. We make the same assumptions about the noise field as we did in Section III, plus two additional assumptions, stated below. To simplify the calculations, let us start with a continuous azimuthal power distribution function $P(\theta)$, bearing in mind that this is the limiting case of a discrete azimuthal power distribution function. Recalling our previous notation, we have

$$P(\theta) = \text{azimuthal power distribution function} \qquad (9.1)$$

$$\Phi(f) = \text{inline horizontal autopower spectrum}$$

$$K(f) = \text{inline horizontal-vertical transfer function}$$

As mentioned above, we will make two additional assumptions about the noise field; these assumptions are not stringent, and they are made in order to allow us to derive simple expressions for the crosspower spectra between spatially separated seismometers. We assume

(1) Noise from each direction propagates in plane waves

(2) Let h and h' be the outputs of two horizontal seismometers located at points A and B respectively; let $\vec{r}$ be the vector beginning at A and ending at B; let both seismometers be pointing in the direction $\vec{r}$. Then for a single noise source generating waves traveling in the direction $\vec{r}$, we have

$$h' = q \circledast h, \text{ the Fourier transform of } q(t) \text{ being} \qquad (9.2)$$

$$Q(f) = \exp(2\pi i r k(f)), \text{ where } r = |\vec{r}|, \qquad (9.3)$$

$$k = \text{wave number}$$

$$\left| \text{See Laster et al} \right|$$

Assumption (1) requires that the distance to the noise sources not be small relative to the array dimensions, while assumption (2) demands that inline horizontal motions at two separated points be related by a transfer function $Q(f)$ which is pure phase shift at all frequencies.

C. CALCULATION OF AUTOPOWER AND CROSSPOWER SPECTRA

Returning to our array of seismometers $v, h_1, \ldots, h_M$, let us derive formulas for

$$\Phi_{mn}, \ \Phi_{vn}, \ \text{etc.}$$

Let us first compute the crosspower spectrum between h_m and h_n, horizontal seismometers located at $P_m = (r_m, \rho_m)$ and $P_n = (r_n, \rho_n)$ respectively, and having orientations ψ_m and ψ_n respectively.

Let θ be an angle, and let λ_m and λ_n be the components of horizontal motion in the direction θ at P_m and P_n respectively, <u>due to noise from a single source at azimuth θ</u>, and temporarily let h_m and h_n denote the contribution from this single source to the seismometer outputs.

Then

$$h_m = -\cos(\theta - \psi_m)\, \lambda_m \tag{9.4}$$

$$h_n = -\cos(\theta - \psi_n)\, \lambda_n$$

and, by assumptions (1) and (2),

$$\lambda_m = q \circledast \lambda_n \tag{9.5}$$

where the Fourier transform of $q(t)$ is

$$Q(f) = \exp\left| 2\pi\, i\, k(f)\, (r_m \cos(\theta - \rho_m) - r_n \cos(\theta - \rho_n)) \right| \tag{9.6}$$

Therefore, for a single wave from direction θ, we have, by (9.4) - (9.6) that

$$\Phi_{mn} = \cos(\theta - \psi_m) \cos(\theta - \psi_n) \, \Phi_{\lambda_m \lambda_n} \tag{9.7}$$

$$= \cos(\theta - \psi_m) \cos(\theta - \psi_n) \, Q \, \Phi_{\lambda_n \lambda_n}$$

$$= \cos(\theta - \psi_m) \cos(\theta - \psi_n) \exp\left| 2\pi i k \left(r_m \cos(\theta - \rho_m) \right.\right.$$

$$\left.\left. - r_n \cos(\theta - \rho_n) \right) \right| \, \Phi_{\lambda_n \lambda_n}$$

$$\left| \text{Crosspower spectrum for single source} \right|$$

To obtain the crosspower spectrum for the sum of noise from all sources, we set $\Phi_{\lambda_n \lambda_n} = P(\theta) \cdot \Phi$ in (9.7) and integrate over all values of θ from $-\pi$ to π.

Therefore the crosspower spectrum between h_m and h_n in a single-mode noise field with azimuthal power distribution function $P(\theta)$ is

$$\Phi_{mn} = \Phi \int_{-\pi}^{\pi} \cos(\theta - \psi_m) \cos(\theta - \psi_n) \exp\left| 2\pi i k \left(r_m \cos(\theta - \rho_m) \right.\right. \tag{9.8}$$

$$\left.\left. - r_n \cos(\theta - \rho_n) \right) \right| P(\theta) \, d\theta$$

In case $m = n$, (9.8) reduces to

$$\Phi_{mm} = \Phi \int_{-\pi}^{\pi} \cos^2(\theta - \psi_m) \, P(\theta) \, d\theta \tag{9.9}$$

$$\left| \text{Cf. Eq. 3.10} \right|$$

A similar computation yields

$$\Phi_{vm} = -K \Phi \int_{-\pi}^{\pi} \cos(\theta - \psi_m) \exp \left| 2\pi i k r_m \cos(\theta - \rho_m) \right| P(\theta) d\theta \quad (9.10)$$

where $K(f)$ is the horizontal-vertical transfer function. (See Eq. 9.1).

. As before (Eq. 3.12),

$$\Phi_{vv} = |K|^2 \Phi \quad (9.11)$$

D. PREDICTION FILTERS

The optimum prediction filter transfer functions G_m, $m = 1, 2, \ldots, M$, are determined by the system of equations (Burg, 1964)

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & \cdots & \Phi_{1M} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \Phi_{M1} & \cdots & & \Phi_{MM} \end{bmatrix} \begin{bmatrix} G_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ G_M \end{bmatrix} = \begin{bmatrix} \Phi_{v1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \Phi_{vM} \end{bmatrix} \quad (9.12)$$

where the Φ_{mn}, Φ_{vm} are given in Equations (9.8) - (9.10).

BLANK PAGE

SECTION X

ISOTROPIC SINGLE-MODE NOISE

In this section we shall consider the theoretical capability of particular arrays of spatially separated horizontal seismometers for predicting a vertical component of single-mode isotropic noise. Array geometries will be specified by use of the parameters r_m, ρ_m, ψ_m, and use will be made of formulas (9.8) - (9.12).

A. INTEGRAL REPRESENTATIONS OF BESSEL FUNCTIONS

In order to simplify the expressions which will be obtained in this section, the following integrals are useful. Watson (p. 41) gives the following form for Poisson's integral representation of the Bessel function of first kind and of order ν.

$$J_\nu(z) = \frac{2(z/2)^\nu}{\Gamma(\nu+\frac{1}{2}) \, \Gamma(\frac{1}{2})} \int_0^{\pi/2} \cos(z \cos \theta) \sin^{2\nu} \theta \, d\theta \qquad (10.1)$$

Noting that

$$\Gamma(\frac{1}{2}) = \sqrt{\pi} \quad \text{and} \quad \Gamma(3/2) = \sqrt{\pi}/2,$$

$$J_0(z) = \frac{2}{\pi} \int_0^{\pi/2} \cos(z \cos \theta) \, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(z \cos \theta) \, d\theta \qquad (10.2)$$

$$(10.3)$$

$$\frac{J_1(z)}{z} = \frac{2}{\pi} \int_0^{\pi/2} \cos(z \cos \theta) \sin^2 \theta \, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(z \cos \theta) \sin^2 \theta \, d\theta$$

Webster (p. 322) gives the recurrence relation

$$\frac{J_1(z)}{z} = \frac{1}{2} \left[J_0(z) + J_2(z) \right]$$

Hence,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(z\cos\theta)\sin^2\theta\,d\theta = \frac{1}{2}\left[J_0(z) + J_2(z)\right] \qquad (10.4)$$

Subtracting (10.4) from (10.2), we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(z\cos\theta)\cos^2\theta\,d\theta = \frac{1}{2}\left[J_0(z) - J_2(z)\right] \qquad (10.5)$$

Webster (p. 322) also gives the relation

$$\frac{d}{dz}\cdot J_0(z) = -J_1(z)$$

Hence, differentiation of (10.2) yields

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\theta\,\sin(z\cos\theta)\,d\theta = J_1(z)$$

B. 2-CHANNEL SYSTEM: SEPARATED VERTICAL AND HORIZONTAL

Recall from Section III that no prediction of a vertical component from a horizontal component located at the same point is possible in isotropic noise. This fact is one of the most severe limitations on the potential use of two-component seismometers at point location, since the noise at many recording stations appears to be nearly isotropic.

However, a portion of noise on a vertical seismometer can be predicted from a spatially separated horizontal seismometer, even in isotropic noise; it is this fact that motivates the study of arrays of the general type discussed in the last section.

Let us begin by examining the prediction capability of the following system in single-mode isotropic noise:

SYSTEM 0: Vertical seismometer v located at the origin O, and a single horizontal seismometer h at distance r from the origin, with azimuth $\rho = 0$, and orientation ψ. (Figure X-1)

Since the noise is isotropic, we substitute $P(\theta) = \frac{1}{2\pi}$ in formula (9.10) to obtain Φ_{vh}. We get

$$\Phi_{vh} = \frac{-K\Phi}{2\pi} \int_{-\pi}^{\pi} \cos(\theta - \psi)\, e^{i2\pi rk\cos\theta}\, d\theta \qquad (10.7)$$

$$= \frac{-iK\Phi}{2\pi} \int_{-\pi}^{\pi} \cos\theta\, \cos\psi\, \sin(2\pi rk\cos\theta)\, d\theta$$

$$= -iK\Phi \cdot \cos\psi \cdot J_1(2\pi rk)$$

Now, by (9.9) and (9.10) we have

$$\Phi_{hh} = \frac{\Phi}{2\pi} \int_{-\pi}^{\pi} \cos^2(\theta - \psi)\, d\theta = \frac{1}{2}\Phi \qquad (10.8)$$

$$\Phi_{vv} = |K|^2 \Phi$$

Therefore the coherence between v and h for system 0 in a single-mode isotropic noise is

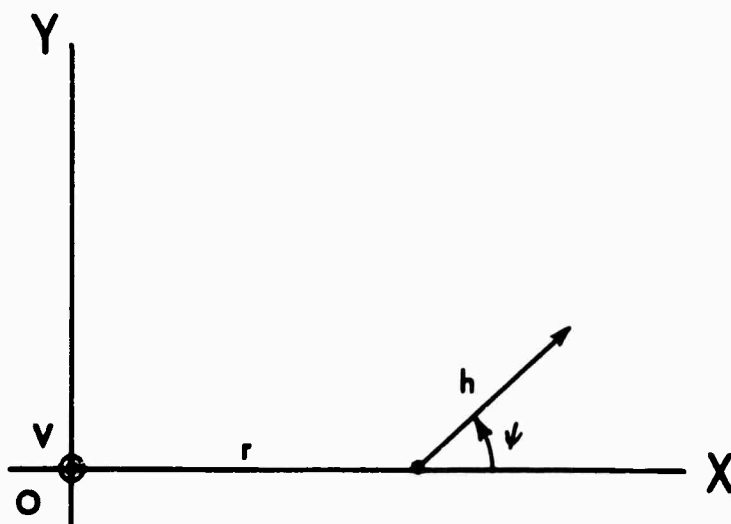$$C = \frac{|\Phi_{vh}|^2}{\Phi_{vv}\Phi_{hh}} = 2\cos^2\psi \left[J_1(2\pi rk)\right]^2$$

Figure X-1. Geometry of System 0

Notice in (10.9) the dependence of coherence on ψ, the horizontal seismometer orientation angle. For $\psi = \pm \dfrac{\pi}{2}$, we have $C = 0$; and for $\psi = 0$, π we have the maximum coherence,

$$C = 2 \left| J_1 (2\pi rk) \right|^2 \qquad (10.10)$$

Let us give the name System 1 to the special instance of System 0 when $\psi = 0$ (Figure X-2). In Table X-1, the prediction error $1-C$ for System 1, where C is calculated by (10.8), is given as a function of $2\pi rk$. With this two-channel system, we get a prediction error as low as 0.324, for $2\pi rk = 1.8$.

C. MULTIPLE-HORIZONTAL ARRAYS

Since part of the vertical component of single-mode isotropic noise can be predicted by a single spatially separated horizontal seismometer, one should expect that very good performance might be achieved by implementing multichannel prediction using a large enough number of additional horizontals, arranged in a suitable geometry. To investigate this possibility, let us consider the arrays shown in Figure X-2. The array parameters are:

$$(10.11)$$

<u>SYSTEM 1.</u> $h_1: r_1 = r, \ \rho_1 = \psi_1 = 0$

<u>SYSTEM 2.</u> $h_m: r_m = r, \ \rho_m = \psi_m = \dfrac{(m-1)\pi}{2}, \ m = 1, 2$

<u>SYSTEM 3.</u> $h_m: r_m = r, \ \rho_m = \psi_m = \dfrac{(m-1)\pi}{3}, \ m = 1, 2, 3$

<u>SYSTEM 4.</u> $h_m: r_m = r, \ \rho_m = \psi_m = m\pi, \ m = 1, 2$

<u>SYSTEM 5.</u> $h_m: r_m = r, \ \rho_m = \psi_m = \dfrac{m\pi}{2}, \ m = 1, 2, 3, 4$

<u>SYSTEM 6.</u> $h_m: r_m = r, \ \rho_m = \psi_m = \dfrac{m\pi}{3}, \ m = 1, 2, 3, 4, 5, 6$

<u>SYSTEM 7.</u> $h_m: r_m = r, \ \rho_m = \psi_m = \dfrac{2m\pi}{3}, \ m = 1, 2, 3$

In all of these systems, the horizontals are place at the same distance, r, from the vertical seismometer location, and the horizontals are oriented so that they point away from the vertical, i.e., $\psi = \rho$.
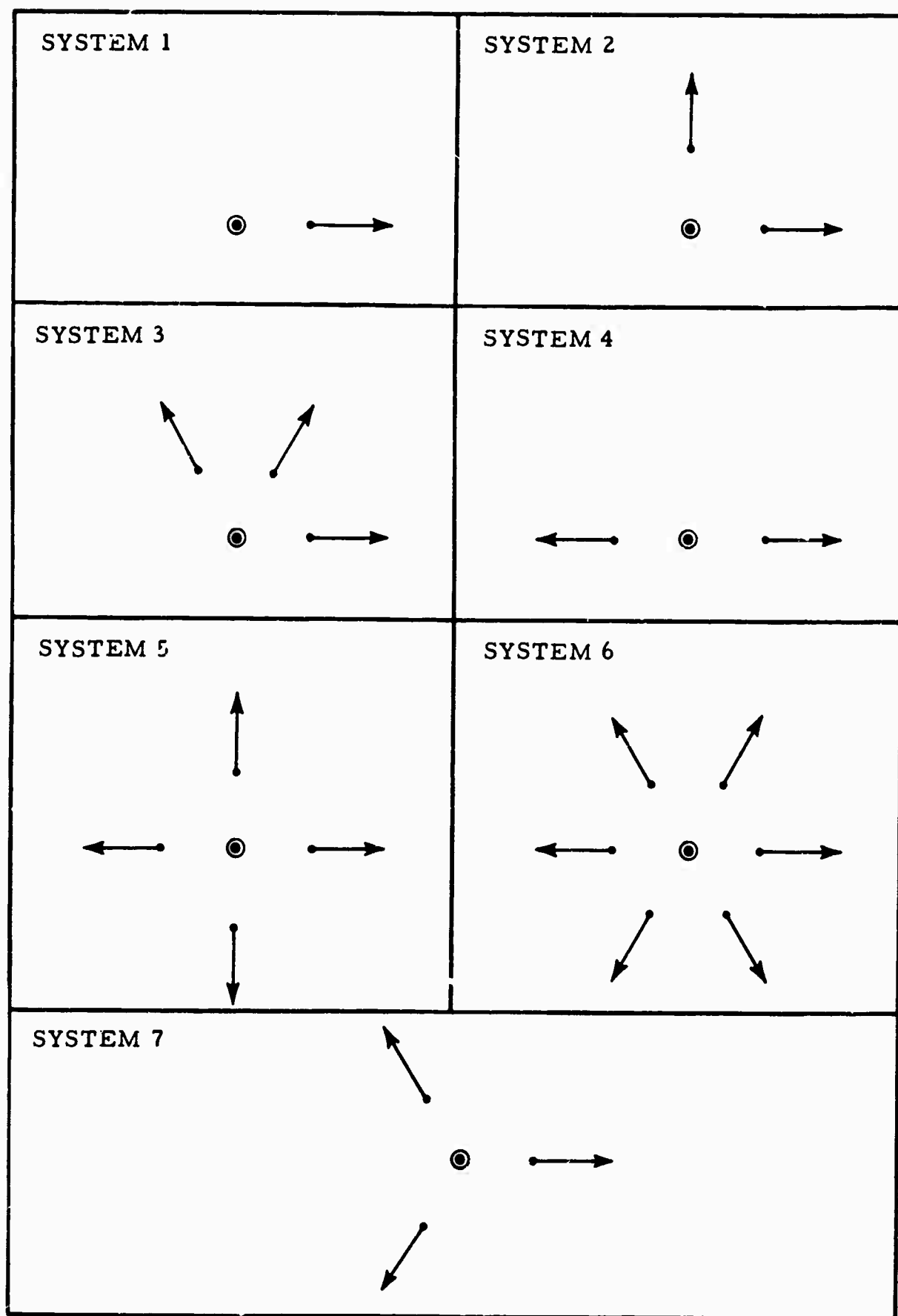
Figure X-2. Array Geometries

Table X-1

PREDICTION ERROR FOR SYSTEM 1

Dimensionless Wavenumber, $2\pi r - k$ (radians)	Prediction Error $(1-C)$
0	1.0
0.2	0.980
0.4	0.923
0.6	0.836
0.8	0.728
1.0	0.613
1.2	0.503
1.4	0.413
1.6	0.350
1.8	0.324
2.0	0.335
2.2	0.382
2.4	0.459
2.6	0.557
2.8	0.664
3.0	0.770
3.2	0.863
3.4	0.936
3.6	0.982
3.8	1.000
4.0	0.991

The decision to restrict our attention to such outward-pointing arrays is motivated by the following fact: let us be given a set of points P_1, P_2, . . ., P_M in the X-Y plane. Then among all arrays V, h_1, h_2, . . ., h_M such that the vertical v is located at O and the horizontals h_m are located at P_m, m = 1, 2, . . ., M, with h_m having arbitrary orientation angle ψ_m, the array which gives the best multi-channel prediction performance in single-mode isotropic noise is the outward-pointing array, that is, the array for which $\psi_m = \rho_m$ for all m. This follows from the fact that maximum coherence between separated vertical and horizontal in single-mode isotropic noise occurs when the horizontal is outward-pointing (Formula 10.9).

Having restricted ourselves to outward-pointing arrays, we further restrict this preliminary investigation to arrays in which all of the horizontals are at equal distance from the vertical because, for an array of outward-pointing horizontals h_m at equal distance from the vertical v, the theoretical optimum prediction filters G_m to predict v from the h_m's in isotropic noise are quite easy to compute. In fact, for each of the Systems 1, 2, 4, 5, 6, and 7 it is obvious that all of the prediction filters are equal, because of symmetry both in the noise field and in the array geometries. Hence, it is necessary to design but one prediction filter for each of these arrays, and in each case apply this filter to the summed outputs of all the horizontals. Therefore, the minimal prediction error for these systems is simply

$$\text{Prediction error} = 1 - C_{vs} \qquad (10.12)$$

where s = summed output of horizontals, and C_{vs} = coherence between v and s.

One can obtain expressions for C_{vs} in terms of Bessel functions for each of the systems 1-7. In the case of system 3, $1-C_{vs}$ may not be the minimal prediction error, since it is possible that the optimum prediction filters for h_0, h_1, h_2 are not identical.

In the case of radially-oriented horizontal-component instruments on a circle of radius r in isotropic noise (with total power Φ), the expressions (9.8) and (9.10) may be simplified, as follows:

In (9.8) let

$$\phi + \alpha_{mn} = \theta - \psi_n = \theta - \rho_n$$

$$\phi - \alpha_{mn} = \theta - \psi_m = \theta - \rho_m$$

$$u = 2\pi k r$$

$$P(\theta) = \Phi/2\pi$$

Thus the crosspower between two horizontals is

$$\Phi_{mn} = \frac{\Phi}{2\pi} \int_{-\pi}^{\pi} \left\{ \cos^2 \alpha_{mn} \sin^2 \phi - \sin^2 \alpha_{mn} \cos^2 \phi \right\} \cdot \qquad (10.13)$$

$$\cdot \cos(2u \sin \alpha_{mn} \cos\phi) \, d\phi$$

$$= \Phi \left\{ \cos(2\alpha_{mn}) J_0(2u \sin \alpha_{mn}) + J_2(2u \sin \alpha_{mn}) \right\} \Big/ 2$$

where $2\alpha_{mn} = \rho_n - \rho_m$ is the angular separation between instruments m and n.

The crosspower between the central vertical and a horizontal is given by substituting in (9.10):

$$\phi = \theta - \psi_m = \theta - \rho_m$$

$$u = 2\pi k r,$$

X-8

giving

$$\Phi_{vm} = \frac{-iK\,\Phi}{2\pi} \int_{-\pi}^{\pi} \cos\phi \, \sin(u\,\cos\phi)\, d\phi \qquad (10.14)$$

$$= -i\,K\,\Phi \cdot J_1(u)$$

The autopower for a horizontal is $\Phi_{mm} = \Phi/2$. This may be verified by setting $\alpha_{mn} = 0$ in (10.13). The autopower for the vertical is $\Phi_{vv} = |K|^2 \Phi$ as given by (10.8).

The coherence between the output of the vertical and the summed outputs of the horizontals is

$$C_{vs} = \frac{\Phi_{vs} \cdot \Phi_{sv}}{\Phi_{vv}\,\Phi_{ss}} \qquad (10.15)$$

where

$$\Phi_{vs} = \sum_{m=1}^{M} \Phi_{vm} \qquad (10.16)$$

$$= -i\,M\,K\,\Phi\,J_1(u)$$

and

$$\Phi_{ss} = \sum_{m=1}^{M} \sum_{n=1}^{M} \Phi_{mn} \qquad (10.17)$$

$$= \Phi \sum_{m=1}^{M} \sum_{n=1}^{M} \left\{ \cos(2\alpha_{mn})\, J_0(2u\,\sin\alpha_{mn}) \right.$$

$$\left. + J_2(2u\,\sin\alpha_{mn}) \right\} \Big/ 2$$

for an array containing M horizontals.

Thus, (10.18)

$$C_{vs} = \frac{2M^2 \left|J_1(u)\right|^2}{\displaystyle\sum_{m=1}^{M}\sum_{n=1}^{M}\left\{\cos(2\alpha_{mn}) J_0(2u \sin \alpha_{mn}) + J_2(2u \sin \alpha_{mn})\right\}}$$

Denoting the coherence for system 1 by C_{vs}^1, the coherence for system 2 by C_{vs}^2, etc., we have:

$$C_{vs}^1 = \left|J_1(u)\right|^2 \Big/ \left(\tfrac{1}{2}\right) \qquad\qquad (10.19)$$

$$C_{vs}^2 = 4\left|J_1(u)\right|^2 \Big/ \left\{1 + J_2\left(\sqrt{2}\,u\right)\right\}$$

$$C_{vs}^3 = 9\left|J_1(u)\right|^2 \Big/ \left\{\tfrac{3}{2} + J_0(u) - \tfrac{1}{2}J_0\left(\sqrt{3}\,u\right) + 2J_2(u) + J_2\left(\sqrt{3}\,u\right)\right\}$$

$$C_{vs}^4 = 4\left|J_1(u)\right|^2 \Big/ \left\{1 - J_0(2u) + J_2(2u)\right\}$$

$$C_{vs}^5 = 16\left|J_1(u)\right|^2 \Big/ \left\{2 + 4J_2\left(\sqrt{2}\,u\right) + 2J_2(2u) - 2J_0(2u)\right\}$$

$$C_{vs}^6 = 36\left|J_1(u)\right|^2 \Big/ \left\{3 + 3J_0(u) + 6J_2(u) - 3J_0\left(\sqrt{3}\,u\right)\right.$$
$$\left. + 6J_2\left(\sqrt{3}\,u\right) + 3J_2(2u) - 3J_0(2u)\right\}$$

$$C_{vs}^7 = 9\left|J_1(u)\right|^2 \Big/ \left\{\tfrac{3}{2} + 3J_2\left(\sqrt{3}\,u\right) - \tfrac{3}{2}J_0\left(\sqrt{3}\,u\right)\right\}$$

The prediction errors $(1-C_{vs})$ for the seven systems are illustrated in Figures X-3 through X-9 by the curves labeled "P.E." In addition to the prediction error, there are certain functions which determine the usefulness of each array. It is not sufficient that the prediction error be low, if the power response Φ_{ss} is also low, since the optimum prediction filter response is inversely proportional to Φ_{ss} and it is desirable to avoid undue amplification of uncorrelated noise appearing on the horizonal outputs.
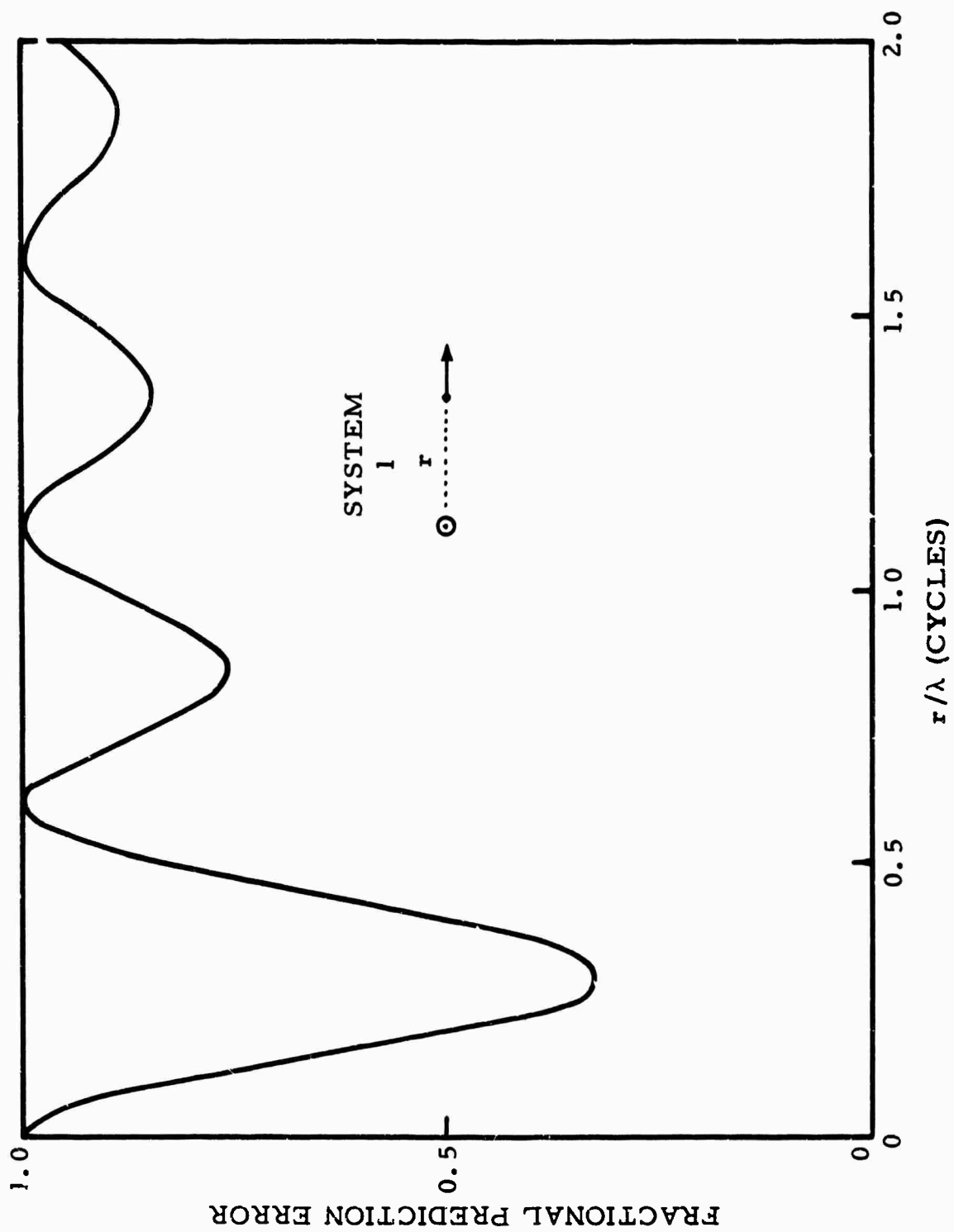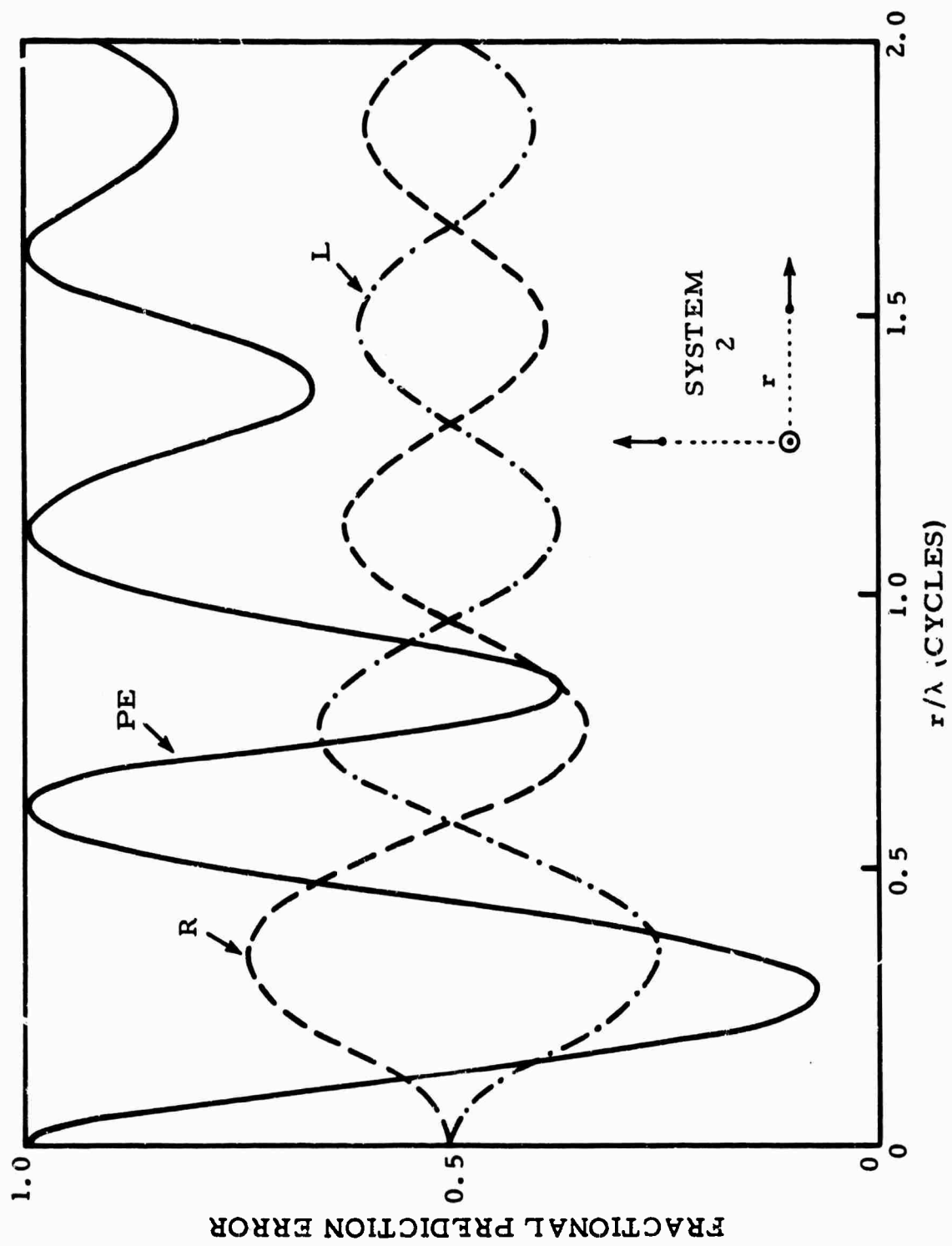
Figure X-3. Prediction Error for System 1

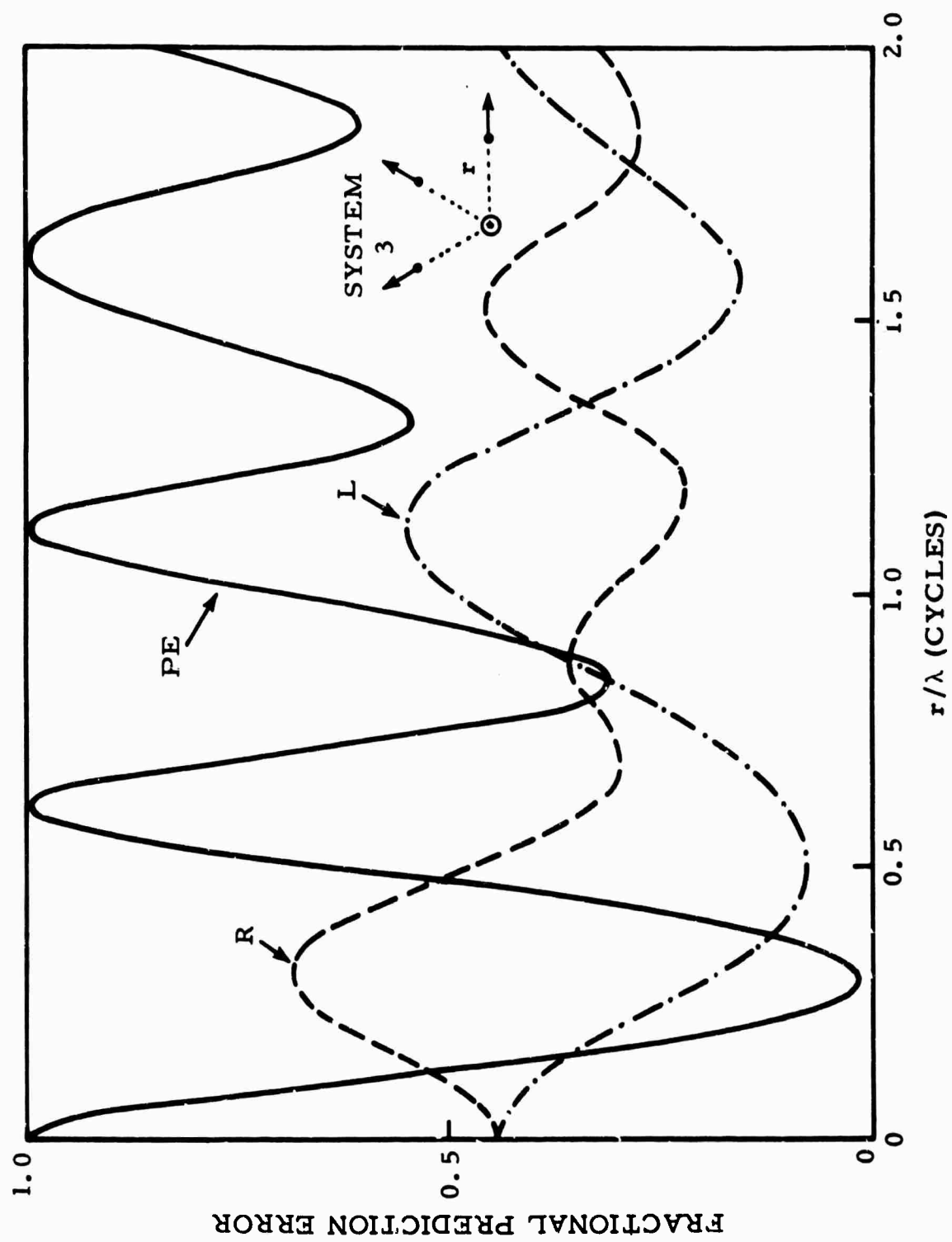Figure X-4. Response Functions for System 2
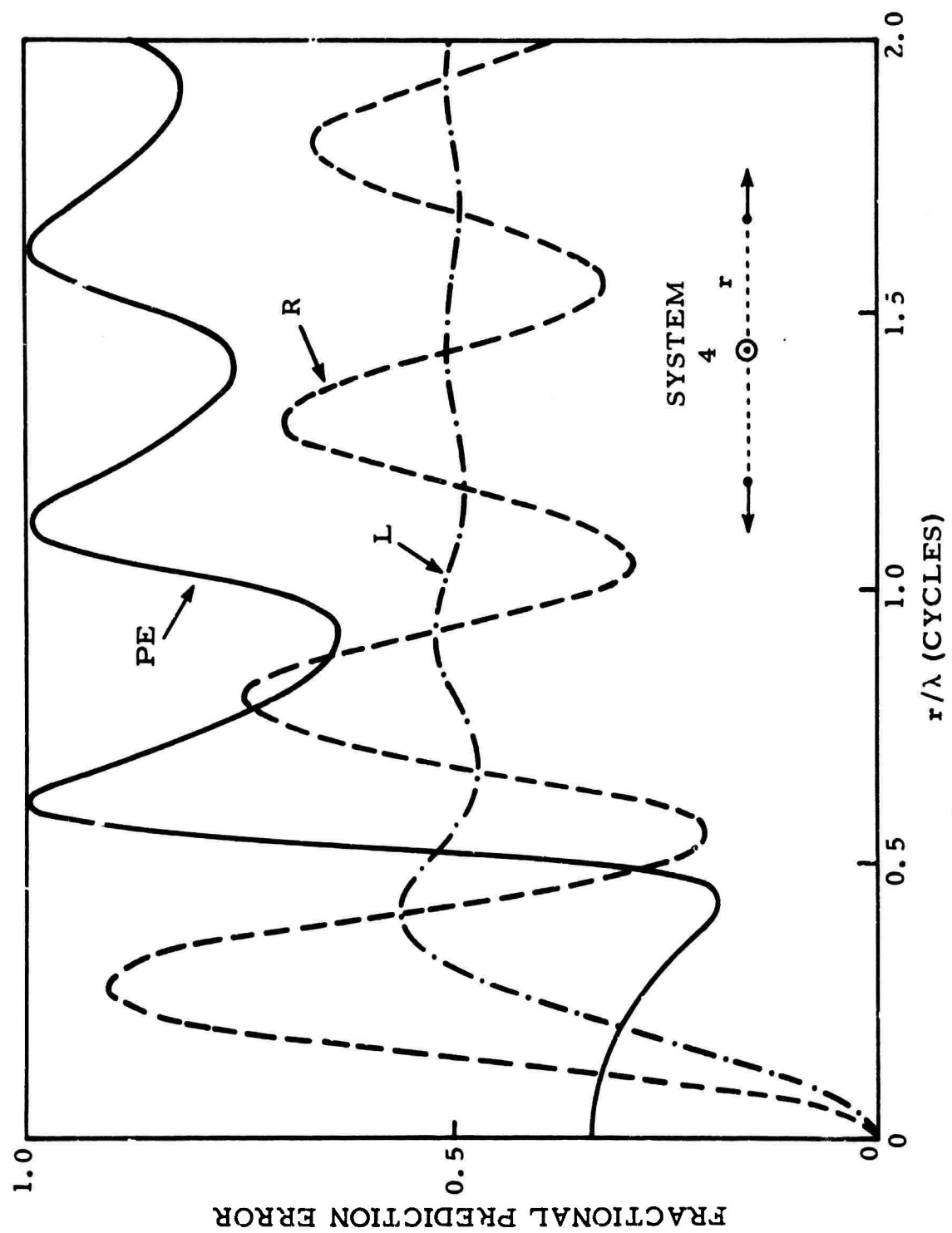
Figure X-5. Response Functions for System 3
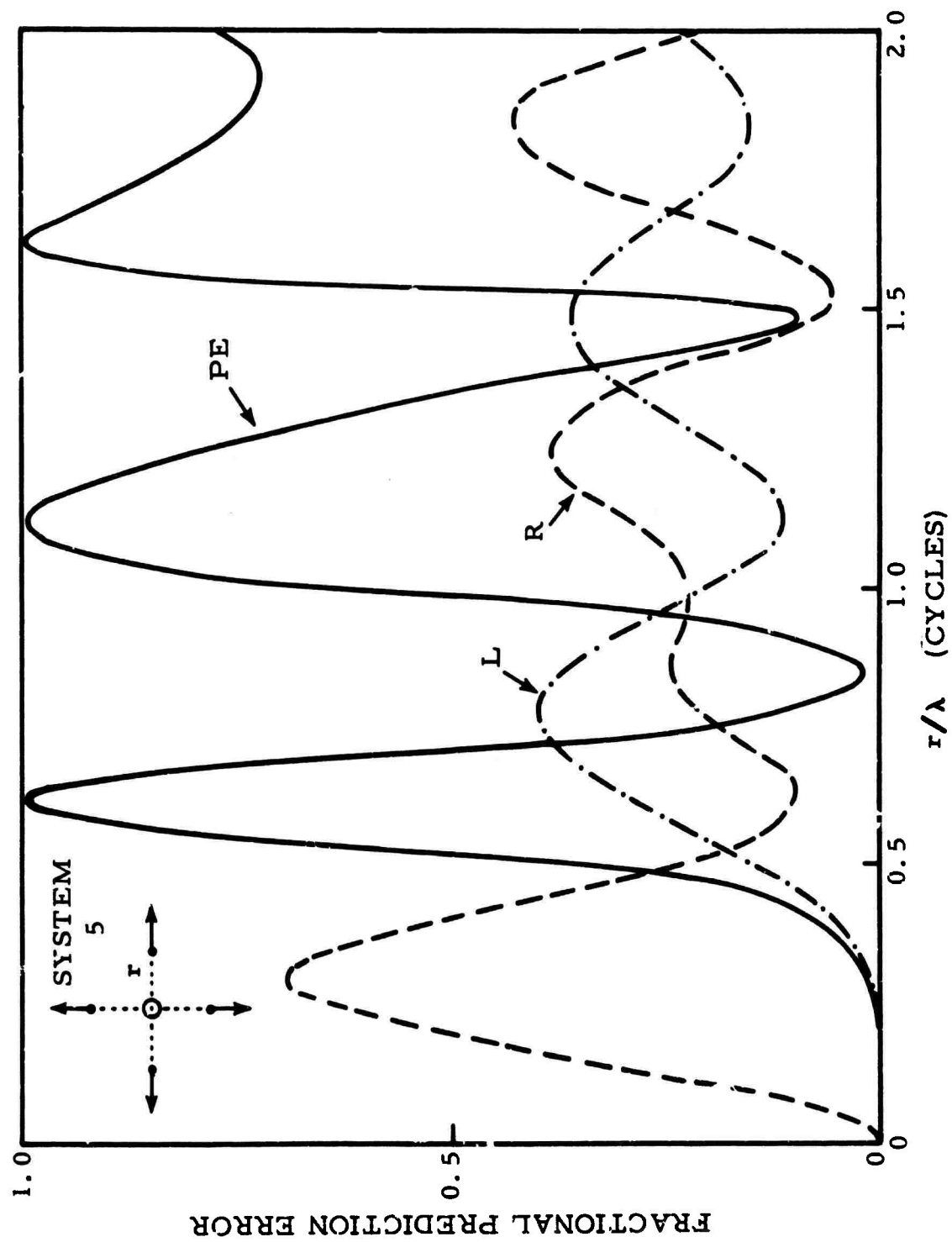
Figure X-6. Response Functions for System 4
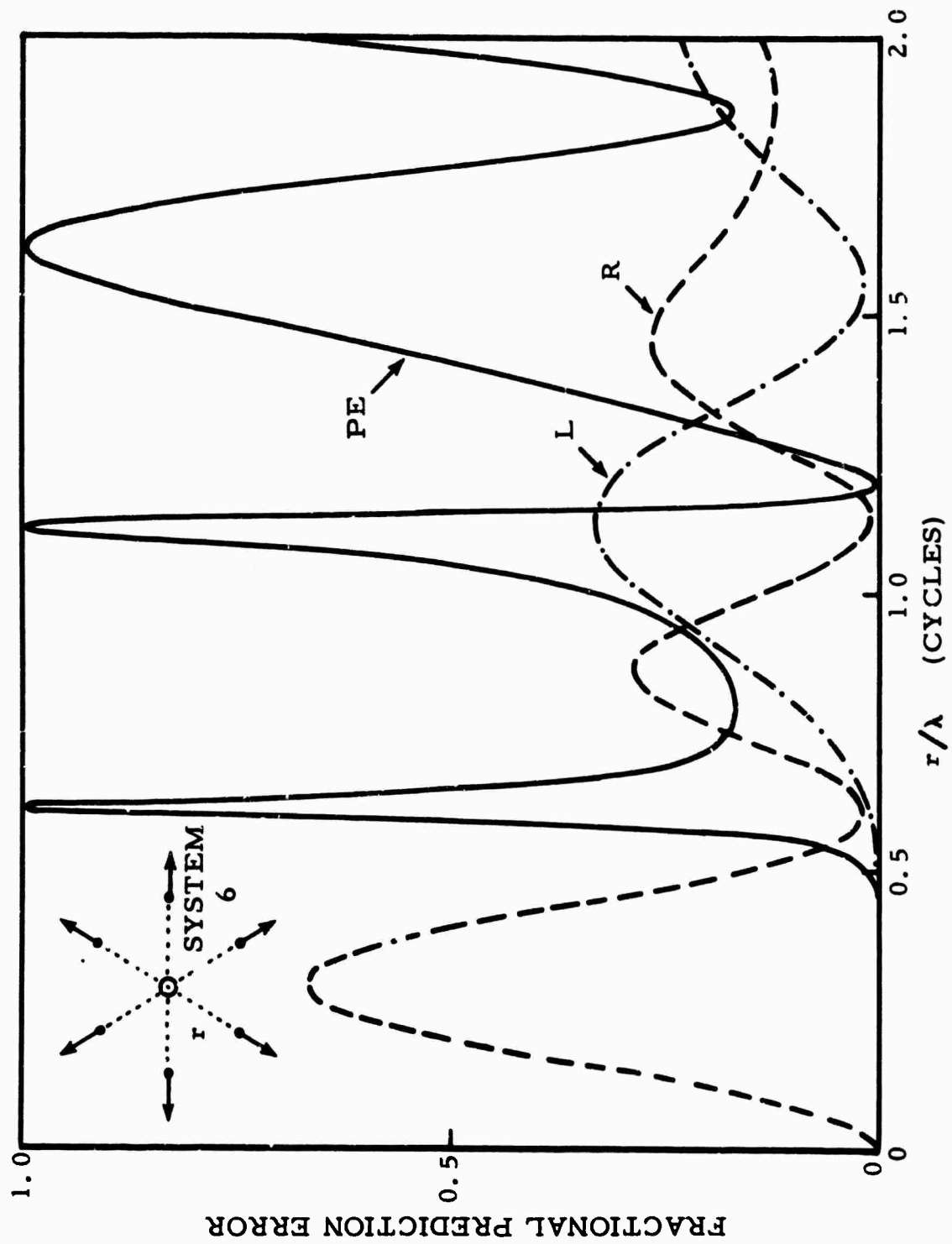
Figure X-7. Response Functions for System 5
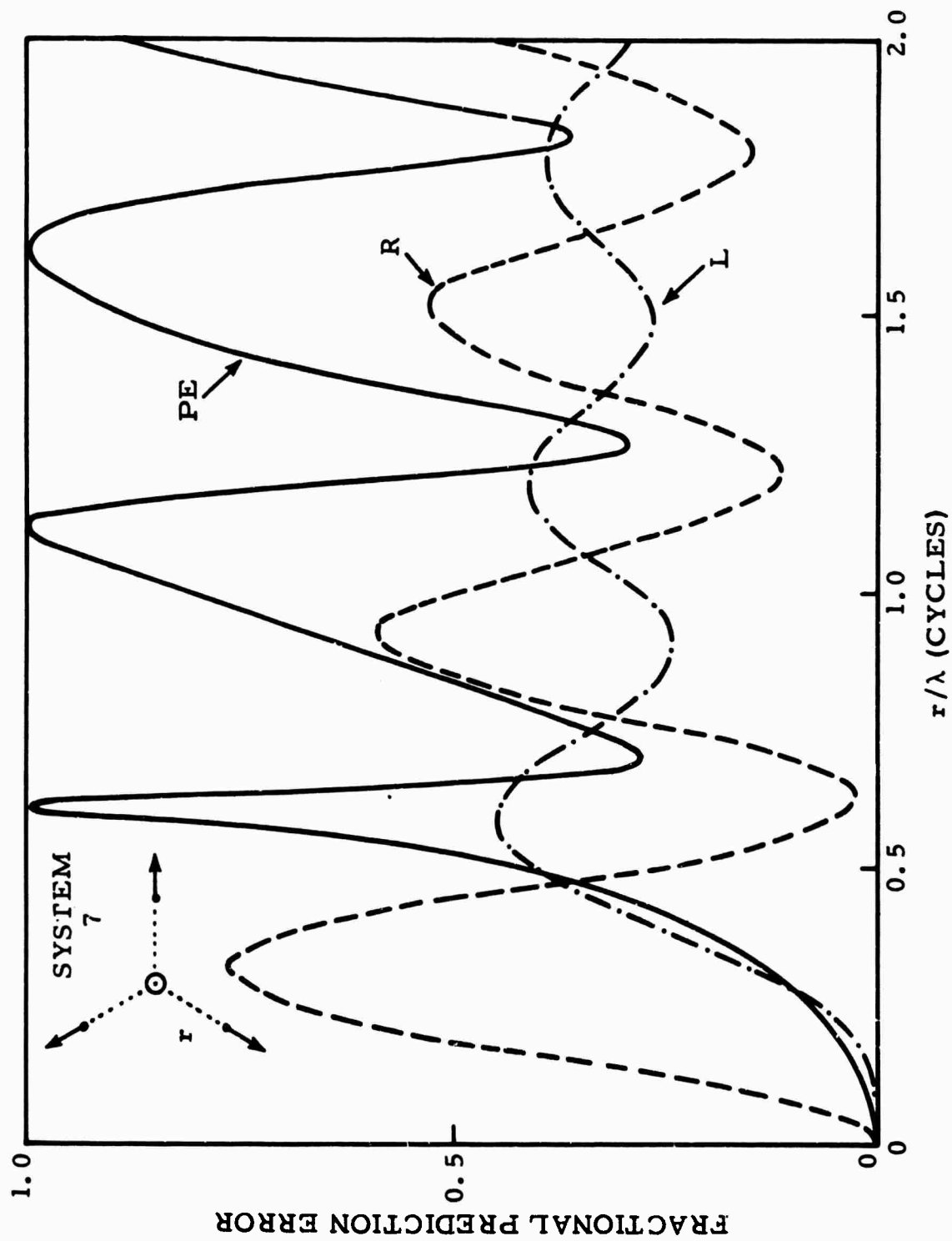
Figure X-8. Response Functions for System 6

Figure X-9. Response Functions for System 7

A measure of the amplification of Rayleigh noise relative to uncorrelated noise is shown by the curves marked "R" in Figures X-3 through X-9. The function plotted is the ratio of Rayleigh noise power in the <u>average</u> of the M horizontal outputs (Φ_{ss}/M^2) to the Rayleigh noise power in the output of a single horizontal ($\Phi/2$). The corresponding ratio for uncorrelated noise is $1/M$.

D. LOVE-WAVE RESPONSES

Another important consideration is the response of the system to Love waves. Since the vertical instrument is insensitive to Love waves, any Love energy which appears in the average of the horizontals will lead to erroneous prediction of noise on the vertical. It is therefore desirable that the system be as insensitive as possible to Love waves. In Figures X-3 through X-9, the curves labeled "L" show the ratio of Love noise power in the average of the M horizontal outputs (Λ_{ss}/M^2) to the Love noise power in the output of a single horizontal.

The Love-wave responses are computed as follows: For our purposes, we may consider a Love-wave to be identical to a Rayleigh wave except that the motion is perpendicular to (instead of inline with) the direction of propagation and there is no vertical component of motion. Assume a uniform power distribution $\Lambda/2\pi$. Since there is no vertical component,

$$\Lambda_{vm} = \Lambda_{vv} = 0.$$

Since the horizontal motion is now transverse, the factors $\cos(\theta - \psi_m)$ and $\cos(\theta - \psi_n)$ in equation (9.8) must be replaced by $\sin(\theta - \psi_m)$ and $\sin(\theta - \psi_n)$. By means of the same substitutions as were used to obtain Equation (10.13), we derive the analogous expression

$$\Lambda_{mn} = \frac{\Lambda}{2\pi} \int_{-\pi}^{\pi} \left\{ \cos^2 \alpha_{mn} \cos^2 \phi - \sin^2 \alpha_{mn} \sin^2 \phi \right\} \cos(2u \sin \alpha_{mn} \cos \phi) \, d\phi$$

$$= \Lambda \left\{ \cos(2\alpha_{mn}) J_0(2u \sin \alpha_{mn}) - J_2(2u \sin \alpha_{mn}) \right\} \Big/ 2$$

The power spectrum for the sum of M horizontals is

$$\Lambda_{ss} = \Lambda \sum_{m=1}^{M} \sum_{n=1}^{M} \left\{ \cos(2\alpha_{mn}) J_0(2u \sin \alpha_{mn}) - J_2(2u \sin \alpha_{mn}) \right\} \Big/ 2$$

Rayleigh-wave response functions Φ_{ss}/Φ for the 7 systems are given by the denominators in Equations (10.19). Corresponding Love-wave response functions Λ_{ss}/Λ may be obtained by changing the signs of all terms in the corresponding Rayleigh functions which contain Bessel functions of order 2.

BLANK PAGE

SECTION XI
CONCLUSIONS

Theoretical results obtained in this study indicate that arrays of horizontal-component seismometers should prove to be useful tools for the removal of trapped-mode noise from the outputs of vertical-component instruments. A necessary condition is that the horizontals must be spatially separated from the vertical. In the case of a single-noise mode, arrays such as those studied in Section X should perform best when the array diam r is approximately one-half wavelength. Signal enhancement systems employing only vertical instruments require, in general, array diameters of at least one wavelength. Thus, multicomponent arrays should offer meaningful advantages in terms of land and telemetry requirements.

It is unlikely that difficulties presented by system noise, uncorrelated seismic noise and Love waves should be any more serious than they are in the case of vertical-component arrays. The effect of additional noise modes on multicomponent array performance has not been studied yet. However, it is reasonable to assume that a multiplicity of modes may be dealt with by the application of multichannel filter techniques to the outputs of rings of horizontals. It has been shown that an array consisting of a number of rings of verticals can be useful in the presence of a similar number of noise modes, and there is no known reason for assuming that this usefulness might be a property peculiar to vertical-component arrays.

The results derived theoretically for multicomponent seismomter arrays are quite conducive to optimism. Referring to Figures X-7 and X-8, it can be seen that, if the array dimensions are suitable ($r/\lambda \sim 0.25$), the theoretical prediction error is less than 0.01, the Love wave response function is less than 0.0$\colon$ and the Rayleigh wave response function is approximately 0.7. Thus, it is theoretically possible to predict (and hence remove) more than 99 percent of the Rayleigh noise on the vertical component in the frequency range in which the wavelength is appropriate. The low value of the Love wave response function implies that very little extraneous noise power should be introduced into the system output as a result of Love waves appearing in the outputs of the horizontal instruments. The high value of the Rayleigh wave response function implies that the filter responses need not be unduly large and that uncorrelated noise from the horizontals will not be amplified to a serious degree. The finding that all three response functions take on desirable values in the same range of wavelength is both unexpected and fortuitous.

From the results of the theoretical investigations reported in this report, it is concluded that multicomponent seismometer arrays offer a great deal of promise for signal enhancement applications. It is recommended that experimental investigations of such systems be undertaken as soon as possible.

BIBLIOGRAPHY

Backus, Milo, John Burg, Dick Baldwin and Ed Bryan, 1964, Wideband extraction of mantle P-waves from ambient noise: Geophysics, v. 29, Oct., p. 672-692.

Burg, John P., 1964, Three-dimensional filtering with an array of seismometers: Geophysics, v. 29, Oct., p. 693-713.

Laster, Stan, Joe G. Foreman, and A. Frank Linville, 1965, Theoretical investigation of modal seismograms for a layer over a half-space: Geophysics, v. 30, Aug., p. 571-596.

Lee, Y.W., 1960: Statistical Theory of Communication, John Wiley and Sons, Inc., New York.

Roden, Robert B., 1964, Some theoretical calculations of the performance of a six-element vertical seismometer array: Texas Instruments Incorporated, Unpublished Report, Jun.

Texas Instruments Incorporated, 1965, Potential use of 3-component seismometers: Appendix A of Research on Teleseismic Signal Extraction and Analysis, Proposal 126-GD65, Sponsored by AFTAC, May 28.

Watson, G.N., 1944: Treatise on the Theory of Bessel Functions, Cambridge University Press.

Webster, A.G., 1955: Partial Differential Equations of Mathematical Physics, Dover Publications Inc.